Math 201 Homework for Friday, Week 6

Due: Friday, October 8.

PROBLEM 1. Let $\mathcal{P}_n(\mathbb{R})$ be the vector space of polynomials in x of degree at most n with coefficients in $\mathbb{R}$. Define

$$f\colon \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$$

$$p(x) \mapsto \int_0^x p(t)\,dt.$$

Note that this is a definite integral, so there is no constant of integration! One can show f is a linear transformation (but you don't need to do that for this problem).

(a) Find the matrix representing f with respect to the ordered bases $\langle 1, x, x^2\rangle$ for $\mathcal{P}_2(\mathbb{R})$ and $\langle 1, x, x^2, x^3\rangle$ for $\mathcal{P}_3(\mathbb{R})$.

(b) Find the matrix representing f with respect to the ordered bases $\langle 1+x+x^2, x+x^2, x^2\rangle$ for $\mathcal{P}_2(\mathbb{R})$ and $\langle 1+x+x^2+x^3, x+x^2+x^3, x^2+x^3, x^3\rangle$ for $\mathcal{P}_3(\mathbb{R})$.

PROBLEM 2. Let V be a finite-dimensional vector space over F, and let $f\colon V \to V$ be a linear transformation. Prove that f is one-to-one if and only if it is onto.

PROBLEM 3. Let $\mathcal{P}(\mathbb{R})$ be the vector space of polynomials in x with coefficients in $\mathbb{R}$. Define

$$f\colon \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$$

$$p(x) \mapsto \int_0^x p(t)\,dt.$$

and

$$g\colon \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$$

$$p(x) \mapsto p'(x).$$

Again, one can show that f and g are linear transformations, but you don't have to do that here.

(a) Prove that f is one-to-one, but not onto.

(b) Prove that g is onto, but not one-to-one.

(c) What can you say about $f \circ g$ and $g \circ f$?

NOTE: Contrast the situation in this problem with that in problem 2.