Math 201 Homework for Friday, Week 5

Due: Friday, October 1.

PROBLEM 1. For the following functions f:

(i) prove that f is a linear transformation,

(ii) find bases for $\mathcal{N}(f)$ and $\mathcal{R}(f)$, and

(iii) compute the nullity and the rank of f.

(a) $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by $f(x, y, z) = (x - y + z, 2y + z)$.

(b) $f\colon \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ defined by $f(p(x)) = x \cdot p(x) + 2p'(x)$.

(Recall that $\mathcal{P}_n(F)$ denotes the vector space of polynomials with coefficients in F of degree less or equal to n. Here, $p'(x)$ denotes the standard derivative from calculus.)

PROBLEM 2. Let V, W and U be finite-dimensional vector spaces over F, and let $f\colon V \to W$, $g\colon W \to U$ be a linear transformations. One can prove[1] that $g \circ f$ is also a linear transformation.

(a) Show that
$$\operatorname{rank}(g \circ f) \leq \min\{\operatorname{rank}(f), \operatorname{rank}(g)\}.$$

(b) Give an example in which the inequality is strict.

PROBLEM 3. Let V and W be vector spaces over F, and let $f\colon V \to W$ be linear an isomorphism (bijective linear transformation). Let $g\colon W \to V$ be the inverse function to f, that is, g satisfies that $g \circ f = \mathrm{id}_V$ and $f \circ g = \mathrm{id}_W$. Prove that g is a linear transformation.

[1] You don't have to turn that in, but it is a good exercise for you to try.