Math 201 Homework for Friday, Week 4

Due: Friday, September 24.

Problem 1. Find the coordinates of each given vector v with respect to the ordered basis $B = \langle v_1, \ldots, v_n \rangle$ of V. Show your work.

(a) $v = (4, 1)$, $B = \langle (1, 2), (-2, 3) \rangle$, $V = \mathbb{R}^2$.

(b) $v = (4, 1)$, $B = \langle (1, 0), (0, 1) \rangle$, $V = \mathbb{R}^2$.

(c) $v = x^2 + 2x + 3$, $B = \langle 1, (x-1), (x-1)^2 \rangle$, $V = \mathcal{P}_2(\mathbb{R})$.

(d) $v = x^2 + 2x + 3$, $B = \langle 1, x, x^2, x^3 \rangle$, $V = \mathcal{P}_3(\mathbb{R})$.

Problem 2. Let $X = \{1, 2, 3\}$, and consider the vector space of functions

$$\mathbb{R}^X := \{f \colon X \to \mathbb{R}\}.$$

Recall that for $f, g \in \mathbb{R}^X$ and $r \in \mathbb{R}$, the vector space operations are defined as follows:

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (rf)(x) = r(f(x)).$$

Also recall that in order to prove that $f = g$, one would show that $f(i) = g(i)$, for $i = 1, 2, 3$—that's how one shows they are the same function.

The *zero function* is $z \in \mathbb{R}^X$, defined by $z(1) = z(2) = z(3) = 0$. It's the additive identity for the vector space. Also define the three *characteristic functions*: χ_1, χ_2, χ_3 by

$$\chi_i(j) := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

for $i = 1, 2, 3$. Thus, for instance, $\chi_2(1) = \chi_2(3) = 0$, and $\chi_2(2) = 1$. Define $B = \{\chi_1, \chi_2, \chi_3\}$. Show that B is a basis for $\mathbb{R}^X$ by completing the following steps.

(a) (Warm up) Let $f \in \mathbb{R}^X$ be defined by $f(1) = 5$, $f(2) = \pi$, and $f(3) = -7$. Write f as a linear combination of elements of B.

(b) Let g be an arbitrary element of $\mathbb{R}^X$. Show how to write g as a linear combination of elements of B. (Thus, B spans $\mathbb{R}^X$.)

(c) Show that B is a linearly independent set by proving that if

$$a\chi_1 + b\chi_2 + c\chi_3 = z$$

for some $a, b, c \in \mathbb{R}$, then $a = b = c = 0$.

PROBLEM 3. Let V be a vector space over F and B a basis for V. Let $S \subseteq V$.

(a) Prove that if $B \subsetneq S$, then S is linearly dependent.

(b) Prove that if $S \subsetneq B$, then S does not span V.

NOTE 1: The first statement can be read as "a basis is a maximal linearly independent set in V". The second statement reads as "a basis is a minimal spanning set for V."

BONUS (UNGRADED): Prove the converse of both statements. Let B be a subset of V.

(c) Suppose that B is linearly independent and for every S with $B \subsetneq S$, S is linearly dependent. Prove that B is a basis.

(d) Suppose that B spans V and for every S with $S \subsetneq B$, S does not span V. Prove that B is a basis.