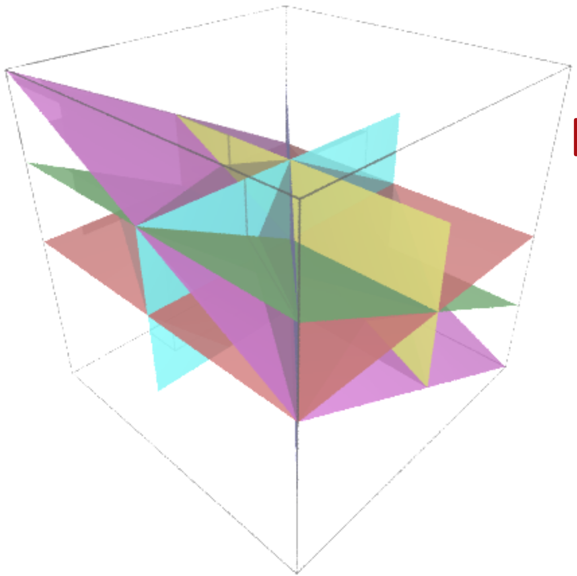

Math 201

Section F03

December 1, 2021

Classroom change for Friday

We meet in Physics 240A on Friday.

Presentations

- Assignments (presentation day, practice talk, peer reviews) link at top of our class homepage

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- Practice with a timer!

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Thus, $\chi(x) = (a_1, \dots, a_n) \cdot x = v_1 \times \cdots \times v_{n-1} \cdot x$, a dot product.

Properties of the cross product

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- The cross product is orthogonal to the subspace spanned by $v_1, \ldots, v_{n-1}$.

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- The length of the cross product is the volume of the parallelepiped spanned by $v_1, \ldots, v_{n-1}$.

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Then

$$v_1\times\cdots\times v_{n-1} = \left(\det A^{(1)}, -\det A^{(2)}, \det A^{(3)}, \ldots, (-1)^{n-1}\det A^{(n)}\right).$$

The cross product in $\mathbb{R}^3$

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$$= (x_2 y_3 - x_3 y_2)\mathbf{i} - (x_1 y_3 - x_3 y_1)\mathbf{j} + (x_1 y_2 - x_2 y_1)\mathbf{k},$$

where $\mathbf{i} = e_1, \mathbf{j} = e_2$, and $\mathbf{k} = e_3$.

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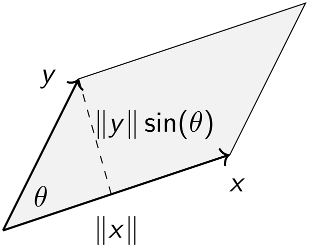
The cross product here is perpendicular to the parallelogram spanned by x and y, and its length is

$$\|x \times y\| = \|x\|\|y\| \sin(\theta)$$



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for some constant d. Plug in any point to find d.