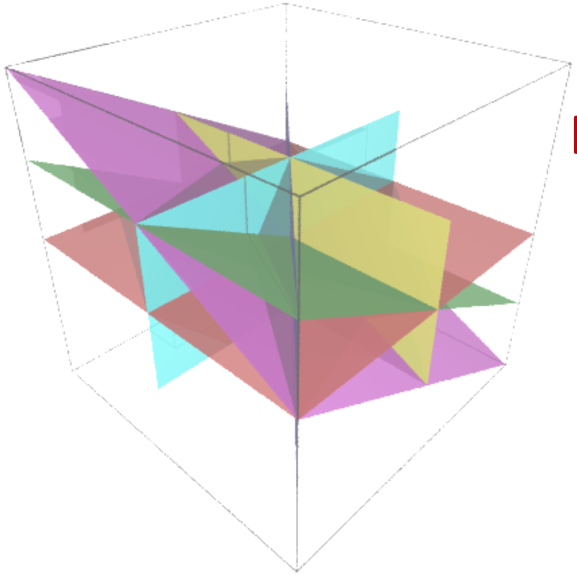

Math 201

Section F03

November 29, 2021

Systems of linear differential equations

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Initial condition:

$$x(0) = c$$

Two-dimensional system

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Rewrite:

$$x'(t) = Ax(t)$$

where

$$x'(t) := \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad x(t) := \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

Main theorem

Theorem. Let A be an $n \times n$ matrix over the real or complex numbers. Then the solution to $x' = Ax$ with initial condition $x(0) = p$ is

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$$= I_n + At + \frac{1}{2} A^2 t^2 + \frac{1}{6} A^3 t^2 + \frac{1}{24} A^4 t^4 + \cdots$$

Computing e^{At}

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$$= Pe^{Dt}P^{-1} = P\,\text{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t})P^{-1}.$$

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$$x_1' = x_2$$
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$$P^{-1}AP = D = \mathrm{diag}(1, -1) \quad \text{where} \quad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

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\begin{aligned}
e^{At} &= Pe^{Dt}P^{-1} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} e^{t} + e^{-t} & e^{t} - e^{-t} \\ e^{t} - e^{-t} & e^{t} + e^{-t} \end{pmatrix}
\end{aligned}
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&= \frac{1}{2} \begin{pmatrix} e^{t}+e^{-t} & e^{t}-e^{-t} \\ e^{t}-e^{-t} & e^{t}+e^{-t} \end{pmatrix}
\end{aligned}
$$

Solution with initial condition $x(0) = (1, 0)$:

$$
\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{t}+e^{-t} & e^{t}-e^{-t} \\ e^{t}-e^{-t} & e^{t}+e^{-t} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{t}+e^{-t} \\ e^{t}-e^{-t} \end{pmatrix}
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$$x_1' = x_2 \qquad x(0) = (x_1(0), x_2(0)) = (0, 1)$$
$$x_2' = x_1$$

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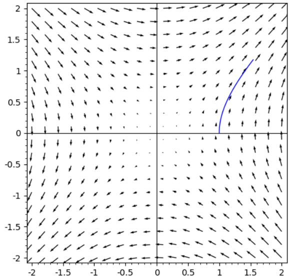

Example 2: frogs and flies

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Diagonalizable over $\mathbb{C}$:

$$P^{-1}AP = D = \mathrm{diag}(i, -i) \quad \text{where} \quad P = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$$

Example 2: frogs and flies

$$\begin{aligned} e^{At} &= Pe^{Dt}P^{-1} \\ &= \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{i\,t} & 0 \\ 0 & e^{-i\,t} \end{pmatrix} \begin{pmatrix} -\frac{1}{2}i & \frac{1}{2} \\ \frac{1}{2}i & \frac{1}{2} \end{pmatrix} \end{aligned}$$

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$$= \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{i\,t} & 0 \\ 0 & e^{-i\,t} \end{pmatrix} \begin{pmatrix} -\frac{1}{2}i & \frac{1}{2} \\ \frac{1}{2}i & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}\,e^{i\,t} + \frac{1}{2}\,e^{-i\,t} & -\frac{1}{2}i\,e^{i\,t} + \frac{1}{2}i\,e^{-i\,t} \\ \frac{1}{2}i\,e^{i\,t} - \frac{1}{2}i\,e^{-i\,t} & \frac{1}{2}\,e^{i\,t} + \frac{1}{2}\,e^{-i\,t} \end{pmatrix}$$

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&= \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}
\end{aligned}
$$

Example 2: frogs and flies

Starting with equal populations of frogs and flies, $x(0) = (1, 1)$,

$$x(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (\cos(t)+\sin(t), -\sin(t)+\cos(t))$$

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