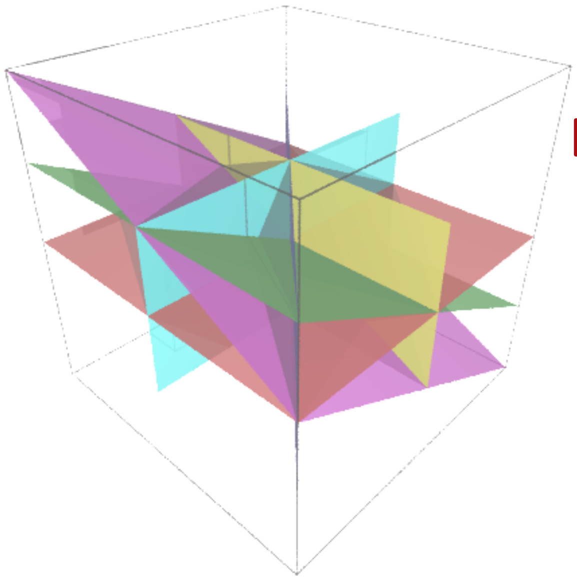

Math 201

Section F03

November 24, 2021

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with scalar multiplication and vector addition defined by

$$\lambda(u, w) = (\lambda u, \lambda w) \quad \text{and} \quad (u, w) + (u', w') = (u + u', w + w'),$$

for all $u, u' \in U$, $w, w' \in W$, and $\lambda \in F$.

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(*internal* versus *external* direct sum)

Review

Recall from last time:

If $S = \{v_1, \ldots, v_k\} \subset V$ is orthonormal and $y \in \mathrm{Span}\, S$, then

$$y = \sum_{i=1}^{k} \langle y, v_j \rangle v_i.$$

Orthogonal complement

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- Hyperplane example

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(d) If $W \subseteq V$ is any subspace, then $(W^\perp)^\perp = W$.

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Corollary. The orthogonal projection u of y onto W is the closest vector in W to y:

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The result now follows from the Pythagorean theorem:

$$\|y - w\|^2 = \cdots$$

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Problem: Approximate $f \in V$ with an element in $\mathrm{Span}(S_n)$.