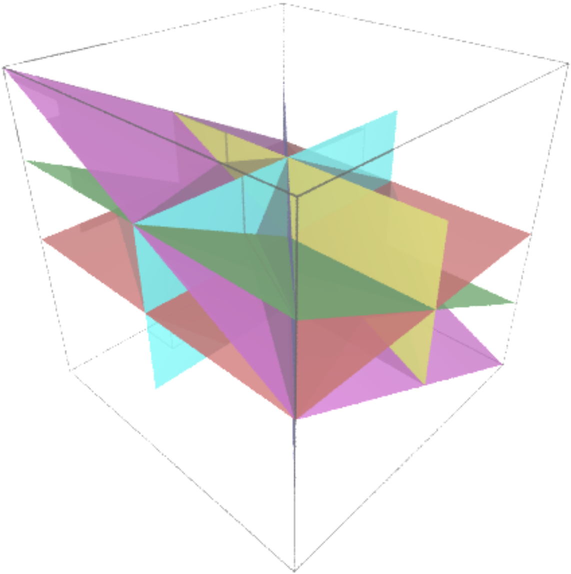

Math 201

Section F03

November 22, 2021

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- $x \in V$ is a *unit vector* if $\|x\| = 1$; equivalently, if $\langle x, x\rangle = 1$

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- The standard basis $\{e_1, \ldots, e_n\}$ for F^n is orthonormal with respect to the standard inner product.
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- $S = \{\cos(x), \sin(x)\}$ is orthogonal in the space of continuous functions $\mathcal{C}(\mathbb{R})$ with respect to the inner product

$$\langle f, g\rangle := \int_0^{2\pi} f(t)g(t)\,dt.$$

Computing coordinates

Proposition. Let $S = \{v_1, \ldots, v_k\}$ be an orthogonal set of nonzero vectors in V, and let $y \in \operatorname{Span} S$. Then

$$y = \sum_{j=1}^{k} \frac{\langle y, v_j \rangle}{\langle v_j, v_j \rangle} v_j = \sum_{j=1}^{k} \frac{\langle y, v_j \rangle}{\|v_j\|^2} v_j.$$

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Corollary 2. Is $S = \{v_1, \ldots, v_k\}$ is an orthogonal set of nonzero vectors in V then S is linearly independent.

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$$u = \frac{1}{\sqrt{2}}(1, 1) \quad \text{and} \quad v = \frac{1}{\sqrt{2}}(1, -1).$$

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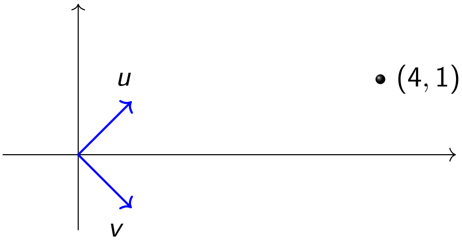

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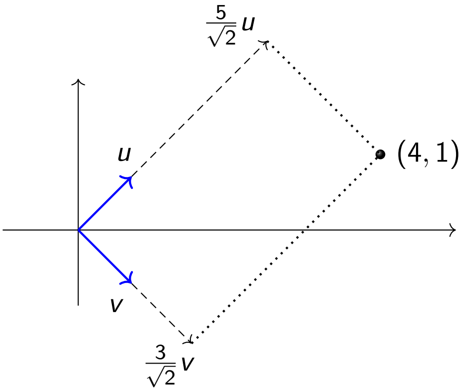

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Algorithm. (Gram-Schmidt orthogonalization)
INPUT: $S = \{w_1, \ldots, w_n\}$, a linearly independent subset of V.

OUTPUT: $S' = \{v_1, \ldots, v_n\}$ an orthogonal set with $\operatorname{Span} S' = \operatorname{Span} S$.

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OUTPUT: $S'' = \left\{ \frac{v_1}{\|v_1\|}, \ldots, \frac{v_n}{\|v_n\|} \right\}$ an orthonormal set with $\operatorname{Span} S' = \operatorname{Span} S$.

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$V = \mathbb{R}_{\leq 1}[x]$ with inner product

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\, dt.$$

Apply Gram-Schmidt to the basis $\{1, x\}$ to get an orthonormal basis.

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Check orthogonality:

$$\langle 1, x - 1/2 \rangle = \int_0^1 (t - 1/2)\,dt = 0.$$

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Orthonormal basis: $\left\{1, \frac{1}{\sqrt{12}}(x - 1/2)\right\}$