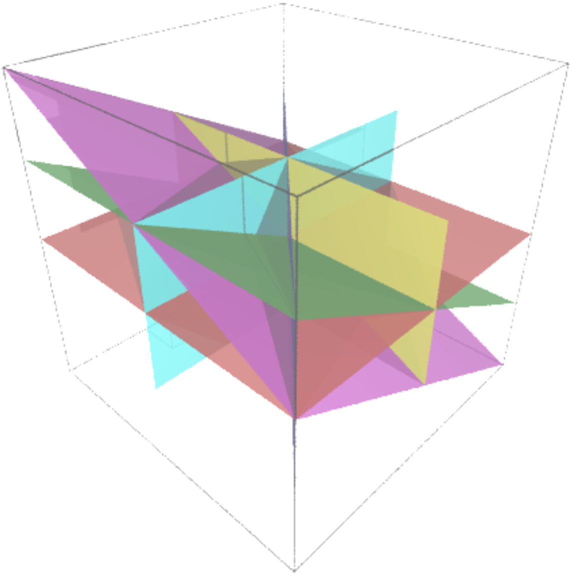

Math 201

Section F03

November 17, 2021

Inner product spaces

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2. conjugate symmetry: $\overline{\langle x, y\rangle} = \langle y, x\rangle$.
3. positive-definiteness: $\langle x, x\rangle \in \mathbb{R}_{\geq 0}$, and $\langle x, x\rangle = 0$ iff $x = 0$.

Properties

Proposition. Let $(V, \langle\ ,\ \rangle)$ be an inner product space over $F = \mathbb{R}$ or $\mathbb{C}$. Then for all $x, y, z \in V$ and $c \in F$,

(a) $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$.

(b) $\langle x, cy\rangle = \overline{c}\langle x, y\rangle$.

(c) $\langle x, 0\rangle = \langle 0, y\rangle = 0$.

(d) If $\langle x, y\rangle = \langle x, z\rangle$ for all $x \in V$, then $y = z$.

Lengths, distances, components, angles

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- $x \in V$ is a *unit vector* if $\|x\| = 1$;

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- $x, y \in V$ are *orthogonal* or *perpendicular* if $\langle x, y \rangle = 0$
- $x \in V$ is a *unit vector* if $\|x\| = 1$; equivalently, if $\langle x, x \rangle = 1$

Examples

$V = \mathbb{R}^n$, $\langle x, y \rangle = x \cdot y$, the usual dot product. Then for $x \in \mathbb{R}^n$,

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

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$V = \mathbb{C}^n$, $\langle x, y \rangle = x \cdot \overline{y}$, the usual dot product on $\mathbb{C}^n$. Then for $z \in \mathbb{C}^n$,

$$\begin{aligned} \|z\| &= \sqrt{z_1 \overline{z_1} + \cdots + z_n \overline{z_n}} \\ &= \sqrt{|z_1|^2 + \cdots + |z_n|^2}. \end{aligned}$$

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If we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via the isomorphism

$$(x_1 + iy_1, \ldots, x_n + iy_n) \to (x_1, y_1, \ldots, x_n, y_n),$$

then the isomorphism preserves norms.