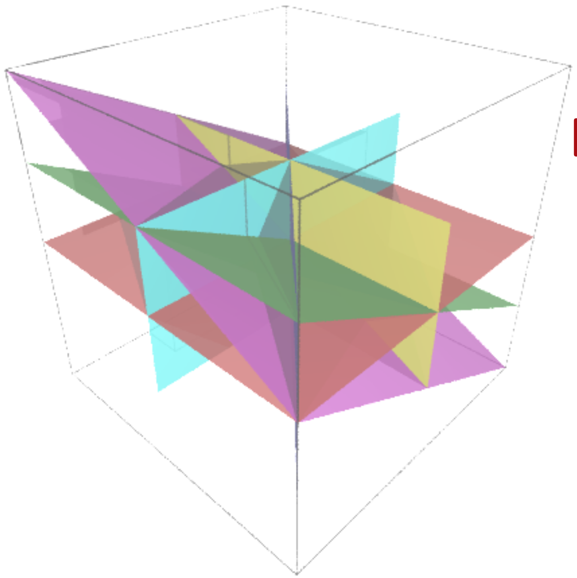

Math 201

Section F03

November 15, 2021

Powers of a matrix

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Therefore,

$$A^\ell = PD^\ell P^{-1} = P \operatorname{diag}(\lambda_1^\ell, \ldots, \lambda_n^\ell)P^{-1}.$$

Walks in graphs

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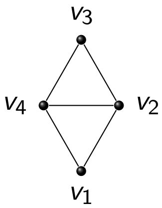

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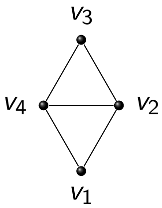

How many are *closed*?

Adjacency matrix

Definition. Let G be a graph with vertices $v_1, \ldots, v_n$. The *adjacency matrix* of G is the $n \times n$ matrix $A = A(G)$ defined by

$$A_{ij} = \begin{cases} 1 & \text{if there is an edge connecting } v_i \text{ and } v_j \\ 0 & \text{otherwise.} \end{cases}$$

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Theorem. Let A be the adjacency matrix for a graph G with vertices $v_1, \ldots, v_n$, and let $\ell \in \mathbb{Z} \geq 0$. Then then number of walks of length ℓ from v_i to v_j is $(A^\ell)_{ij}$.

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(Diamond graph example.)

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For each pair of vertices v and w there are constants $c_1, \ldots, c_n$ such that

$$\#\text{ length } \ell \text{ walks } (v \to w) \quad = \quad c_1\lambda_1^\ell + \cdots + c_n\lambda_n^\ell.$$

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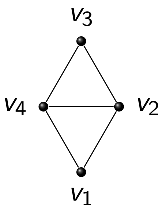
Corollary. Let A be the adjacency matrix of a graph G with n vertices, and let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ be its list of (not necessarily distinct) eigenvalues. Then the number of closed walks in G of length ℓ is $\sum_{i=1}^n \lambda_i^\ell$.

Example

Compute the number of closed walks of length ℓ in the diamond graph:



Extensions

The ideas presented today generalize to directed graphs and to graphs with weighted edges.