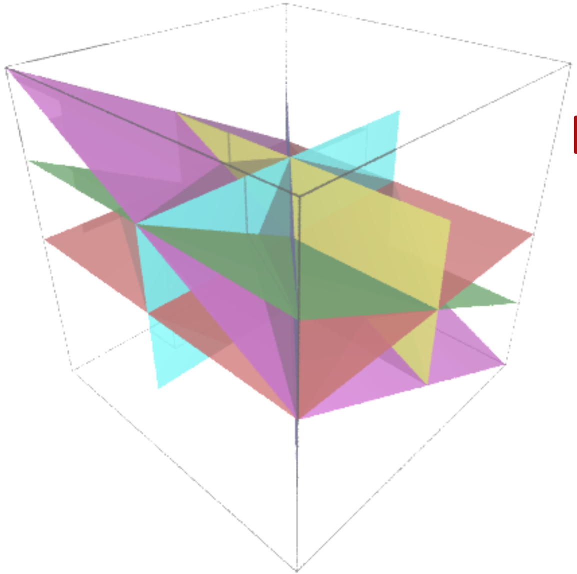

Math 201

Section F03

November 19, 2021

Lengths, distances, components, angles

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- $x \in V$ is a *unit vector* if $\|x\| = 1$; equivalently, if $\langle x, x\rangle = 1$

Examples

$V = \mathbb{R}^n$, $\langle x, y \rangle = x \cdot y$, the usual dot product. Then for $x \in \mathbb{R}^n$,

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$V = \mathbb{C}^n$, $\langle x, y \rangle = x \cdot \overline{y}$, the usual dot product on $\mathbb{C}^n$. Then for $z \in \mathbb{C}^n$,

$$\begin{aligned} \|z\| &= \sqrt{z_1\overline{z_1} + \cdots + z_n\overline{z_n}} \\ &= \sqrt{|z_1|^2 + \cdots + |z_n|^2}. \end{aligned}$$

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If we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via the isomorphism

$$(x_1 + iy_1, \ldots, x_n + iy_n) \to (x_1, y_1, \ldots, x_n, y_n),$$

then the isomorphism preserves norms.

Pythagorean theorem

Proposition. (Pythagorean theorem) Let $(V, \langle\ ,\ \rangle)$ be an inner product space over $F = \mathbb{R}$ or $\mathbb{C}$, and let $x, y \in V$ be perpendicular. Then

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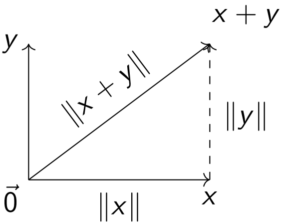

Components and projections

Definition. Let $(V, \langle\ ,\ \rangle)$ be an inner product space over $F = \mathbb{R}$ or $\mathbb{C}$, and let $x, y \in V$ with $y \neq 0$. The *component* of x along y is the scalar

$$c = \frac{\langle x, y\rangle}{\langle y, y\rangle} = \frac{\langle x, y\rangle}{\|y\|^2}.$$

The *orthogonal projection* of x to y is the vector cy.

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(d) Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$.

Distance

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Proposition. For all $x, y, z \in V$,

1. Symmetry: $d(x, y) = d(y, x)$.
2. Positive-definiteness: $d(x, y) \geq 0$, and $d(x, y) = 0$ iff $x = y$.
3. Triangle inequality: $d(x, y) \leq d(x, z) + d(z, y)$.

Angles

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- Cauchy-Schwarz $\Rightarrow -1 \leq \frac{\langle x,y\rangle}{\|x\|\|y\|} \leq 1$
- $\cos(\theta) = \left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|} \right\rangle$