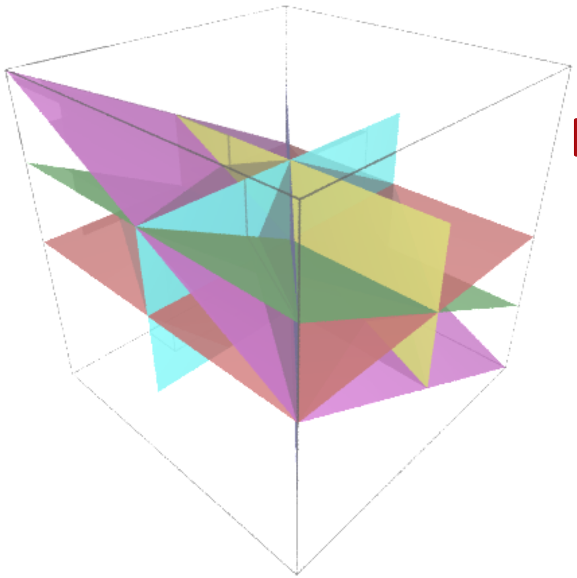

Math 201

Section F03

November 10, 2021

Review

Algorithm for diagonalization:

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- Find the zeros of the characteristic polynomial $p_A(x)$.
- For each eigenvalue λ, compute a basis for the eigenspace E_λ.
- If this process results in finding n eigenvectors, $v_1, \ldots, v_n$, then A is *diagonalizable*. Let P be the matrix with columns $v_1, \ldots, v_n$. Then

  $$P^{-1}AP = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$$

  where $Av_i = \lambda_i$.

Characteristic polynomial

Proposition. Let A, B be $n \times n$ matrices representing a linear function $f\colon V \to V$ with respect to different bases. Then their characteristic polynomials are the same: $p_A(x) = p_B(x)$.

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Therefore, the following definition makes sense:

Definition. Let V be a finite-dimensional vector space. The *characteristic function* of a linear transformation $f\colon V \to V$ is

$$p_f(x) = \det(A - xI_n)$$

where A is the matrix representing f with respect to any basis for V.

Eigenvectors with distinct eigenvalues

Proposition. Let V be any vector space, and let $f: V \to V$ be a linear transformation. Let $v_1, \ldots, v_k \in V$ be eigenvectors for f with corresponding eigenvalues λ_i:

$$f(v_i) = \lambda_i v_i$$

for $i = 1, \ldots, k$. Suppose $\lambda_1, \ldots, \lambda_k$ are *distinct*. Then $v_1, \ldots, v_k$ are linearly independent.

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Corollary. Suppose $\dim V = n$ and $f\colon V \to V$ is a linear transformation. Then if f has n distinct eigenvalues, it is diagonalizable.

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Corollary. Suppose $\dim V = n$ and $f\colon V \to V$ is a linear transformation. Then if f has n distinct eigenvalues, it is diagonalizable. (The converse does not hold.)

Extra material: Cramer's rule

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Cramer's rule: The solution of the $n \times n$ system $Ax = b$ with $b \in F^n$ is

$$x_j = \frac{\det(M_j)}{\det(A)}$$

where $M_j \in M_{n\times n}(F)$ is the matrix formed by replacing the j-th column of A with b.

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$$\mathrm{adj}(A)_{ij} = (-1)^{i+j} \det(A^{ji}).$$

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Corollary of Cramer's rule. If $A \in M_{n\times n}(F)$ is invertible, then

$$A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A)$$

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Corollary. If $A \in M_{n\times n}(F)$ is invertible and $F = \mathbb{R}$ of $F = \mathbb{C}$, then

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Corollary. If $A \in M_{n\times n}(F)$ is invertible and $F = \mathbb{R}$ of $F = \mathbb{C}$, then

1. the solution for the system of equations $Ax = b$ is a continuous function of the entries of A and b, and
2. the entries of A^{-1} are continuous functions of the entries of A.

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