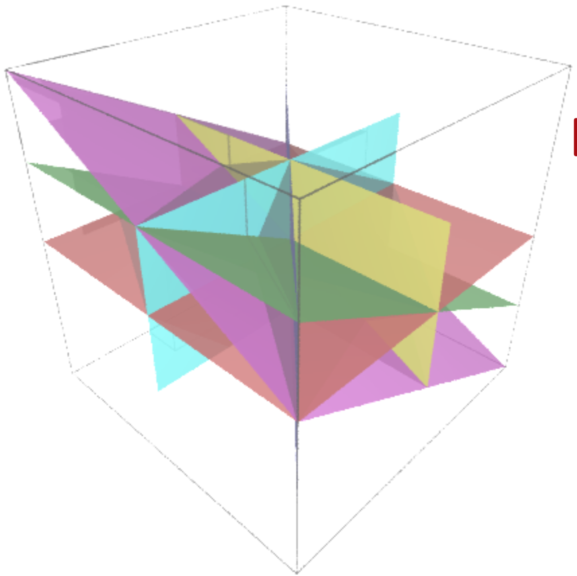

# Math 201

Section F03

November 8, 2021

# New concepts for today

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- Diagonalizability

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- Characteristic polynomial

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- Eigenspace

# Review: eigenvectors and eigenvalues

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A *nonzero* vector $v \in F^n$ is an *eigenvector* for $A \in M_{n\times n}(F)$ with *eigenvalue* $\lambda \in F$ if

$$Av = \lambda v.$$

# Diagonalizability

**Definition.** Let $V$ be an $n$-dimensional vector space. A linear mapping $f: V \to V$ is *diagonalizable* if there exists an ordered basis $\alpha$ of $V$ such that $[f]_\alpha^\alpha = \text{diag}(\lambda_1, \ldots, \lambda_n)$.

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**Proposition.** A linear mapping $f: V \to V$ is diagonalizable if and only if $V$ has a basis consisting solely of eigenvectors for $f$.

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**Definition.** Let $\lambda$ be an eigenvalue of the $n \times n$ matrix $A$ over $F$. Then the *eigenspace for* $\lambda$ is

$$E_\lambda := E(A)_\lambda := \{v \in F^n : Av = \lambda v\} = \ker(A - \lambda I_n v).$$

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# Examples

We will try two examples:

$$A = \begin{pmatrix} 2 & -7 & 3 \\ 0 & -5 & 3 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 1 & 3 \\ 0 & -5 & 3 \\ 0 & 0 & 2 \end{pmatrix}.$$

Only $A$ is diagonalizable.

# First example

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$$= (2-x)(-5-x)(2-x) = -(x-2)^2(x+5).$$

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Find bases for the eigenspaces $E_2 = \ker(A - 2I_3)$ and $E_{-5} = \ker(A + 5I_3)$.

## $E_2 = \ker(A - 2I_3)$

$$A - 2I_3 = \begin{pmatrix} 2 & -7 & 3 \\ 0 & -5 & 3 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -7 & 3 \\ 0 & -7 & 3 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & -3/7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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Basis: $\{(1,0,0),(0,3/7,1)\}$ or, easier, $\{(1,0,0),(0,3,7)\}$.

## $E_{-5} = \ker(A + 5I_3)$

$$A - (-5)I_3 = \begin{pmatrix} 2 & -7 & 3 \\ 0 & -5 & 3 \\ 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -7 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

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$$P^{-1}AP = \mathrm{diag}(2,2,-5)$$

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Basis: $\{(1, 0, 0)\}$ (important: the dimension is 1 this time, not 2).

# $E_{-5} = \ker(B + 5I_3)$

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$B$ is not diagonalizable since $\dim E_2(B) + \dim E_{-5}(B) < 3$.

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We just saw an example illustrating obstacle 1. Next, we will see an example illustrating obstacle 2.

# Ground field not large enough

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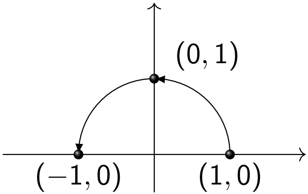

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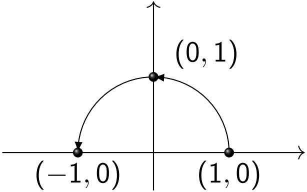


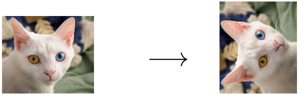

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Fix: work over $\mathbb{C}$.