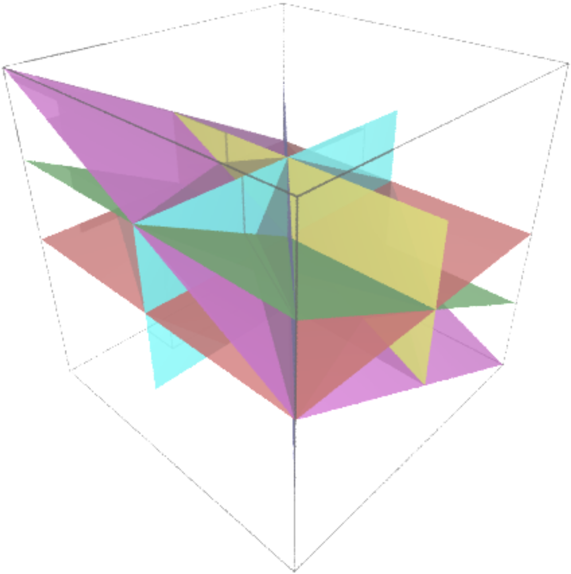

Math 201

Section F03

November 5, 2021

Image of a linear function

Review: The linear function $L_A\colon \mathbb{R}^2 \to \mathbb{R}^2$ scales area by a factor of $\det(A)$.

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Review: The linear function $L_A \colon \mathbb{R}^2 \to \mathbb{R}^2$ scales area by a factor of $\det(A)$. Picture for the case where A has columns $u = (3, -2)$ and $v = (2, 1)$:

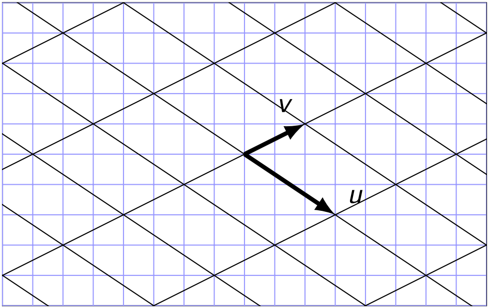

Image of a linear function

Transformation given by the matrix $\begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix}$:

$\longrightarrow$

Image of a linear function

Transformation given by the matrix $\begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix}$:

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Eigenvectors and eigenvalues

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A *nonzero* vector $v \in F^n$ is an *eigenvector* for $A \in M_{n\times n}(F)$ with *eigenvalue* $\lambda \in F$ if

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$$A = \begin{pmatrix} -1 & 2 \\ -6 & 6 \end{pmatrix}$$

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Find the matrix representing f_A with respect to the ordered basis

$$\alpha = \langle (2,3), (1,2) \rangle.$$

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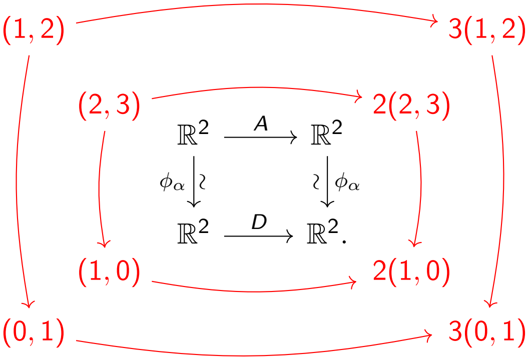

(1, 2)
3(1, 2)
(2, 3)
2(2, 3)
$\mathbb{R}^2$
A
$\mathbb{R}^2$
ϕ_α
$\wr$
$\wr$
ϕ_α
$\mathbb{R}^2$
D
$\mathbb{R}^2$.
(1, 0)
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$$D = P^{-1}AP = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

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$$P^{-1}AP = D,$$

where $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, and we have a commutative diagram

$$\begin{array}{ccc} F^n & \xrightarrow{\ A\ } & F^n \\ {\scriptstyle P^{-1}}\big\downarrow\wr & & \wr\big\downarrow{\scriptstyle P^{-1}} \\ F^n & \xrightarrow{\ D\ } & F^n. \end{array}$$

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To find the eigenvalues of A, solve $\det(A - \lambda I_n) = 0$ for $\lambda \in F$.

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Note: The λ_i are not necessarily distinct: $\dim(\ker(A - \lambda I_n))$ may be greater than 1.

Example

$$A = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

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$\det(A - \lambda I_4) = \lambda(\lambda - 2)(\lambda - 4)^2 = 0 \quad \Leftrightarrow \quad \lambda = 0, 2, 4$

Bases for kernels $A - \lambda I_4$:

$$\begin{aligned} \lambda = 0&: \qquad \{(1,1,1,1)\} \\ \lambda = 2&: \qquad \{(1,0,0,-1)\} \\ \lambda = 4&: \qquad \{(1,0,-2,1),(0,1,-1,0)\} \end{aligned}$$

Example continued

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$$P = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & -2 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix}, \quad P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$