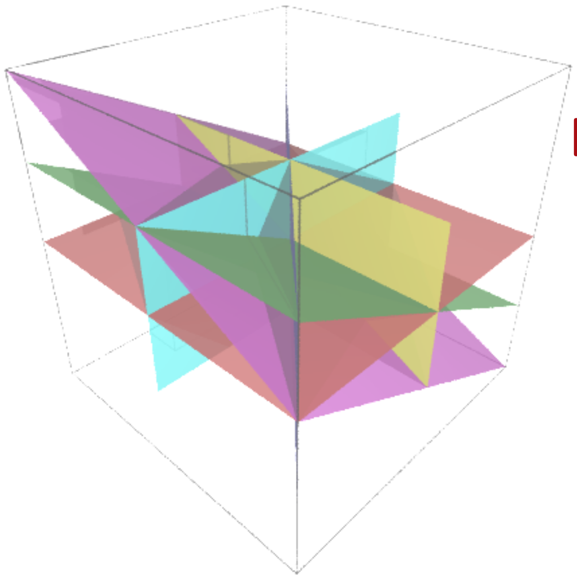

Math 201

Section F03

October 27, 2021

Permutations

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The number permutations of $[n]$ is $|\mathfrak{S}_n| = n!$.

Permutation matrices

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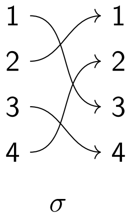
Example. $\sigma \in \mathfrak{S}_4$ with $\sigma(1) = 3$, $\sigma(2) = 1$, $\sigma(3) = 4$, and $\sigma(4) = 2$:



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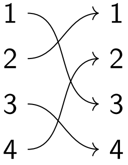

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$$P_\sigma = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

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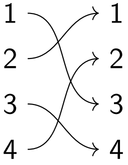

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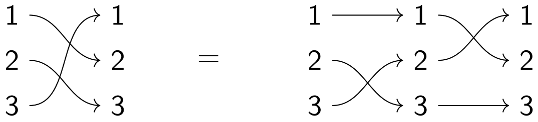

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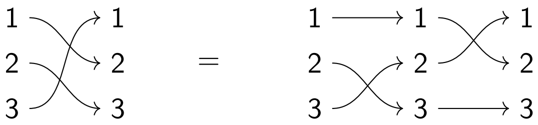

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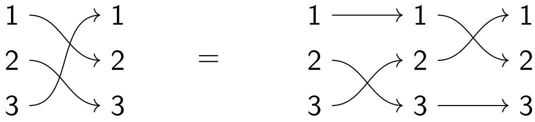

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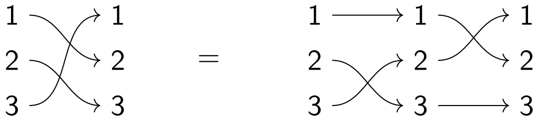

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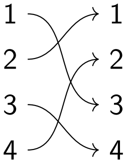
Note: $\sigma \in \mathfrak{S}_n$ is even, resp. odd, if it is the composition of an even, resp. odd, number of transpositions.

Permutation matrices



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$$P_\sigma = \begin{pmatrix} 0 & 0 & \boxed{1} & 0 \\ \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & \boxed{1} & 0 & 0 \end{pmatrix}$$

Think of $\sigma \in \mathfrak{S}_n$ as a (non-attacking) *rook placement* on an $n \times n$ chessboard.

Determinants

Theorem. Let A be an $n \times n$ matrix. Then

$$\det(A) = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sign}(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{n\sigma(n)}.$$

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Think of this formula in terms of rook placements.

Determinants example

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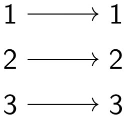

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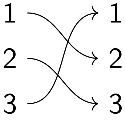

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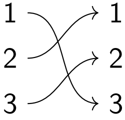

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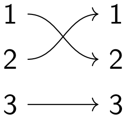

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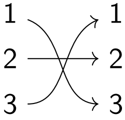

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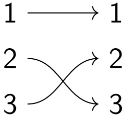

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