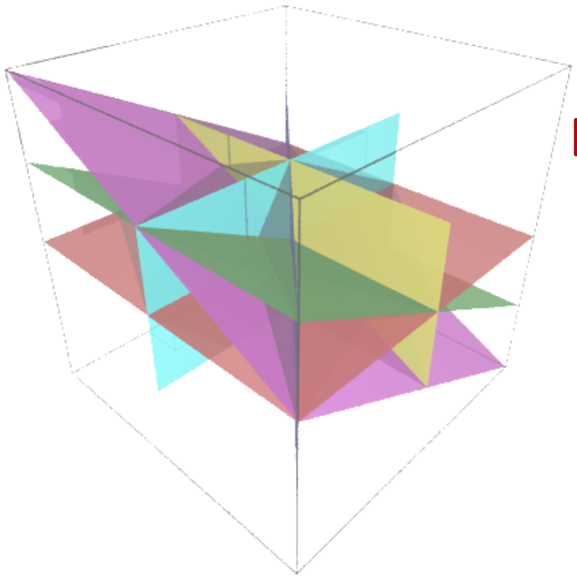

Math 201

Section F03

October 25, 2021

Elementary matrices

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Multiplying $A \in M_{n\times k}(F)$ on the left by E performs that row operation on A:

$$A \xrightarrow{\text{row op}} EA.$$

Examples

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{r_2 \to r_2 - 3r_1} \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =: E_1$$

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Examples

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{r_3 \to r_3 - r_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} =: E_2$$

Examples

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$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -6 & -10 & -10 \\ 0 & 3 & 3 & 3 \end{pmatrix}$$

Examples

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} =: E_3$$

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Examples

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Elementary matrices

If $A \in M_{n \times k}(F)$ has reduced row echelon form $\widetilde{A}$, then there exists a sequence of elementary matrices $E_1, \ldots, E_\ell$ such that

$$\widetilde{A} = E_\ell \cdots E_2 E_1 A.$$

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Definition. The *transpose* of $A \in M_{m\times n}(F)$ is $A^t \in M_{n\times m}(F)$ where

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$$\begin{pmatrix} 5 & 2 \\ 1 & 3 \end{pmatrix}^t = \begin{pmatrix} 5 & 1 \\ 2 & 3 \end{pmatrix}$$

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2. Row scaling by λ: $E^t = E$, and $\det(E) = \det(E^t) = \lambda$.
3. Add λr_i to r_j in I_n: The E^t comes from I_n by adding row λr_j to r_i. So $\det(E) = \det(E^t) = \det(I_n) = 1$.

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Corollary. The determinant is a multilinear, alternating, normalized function of the *columns* of a square matrix.

Example

$$\det \begin{pmatrix} 1 & -1 & -1 & -1 \\ -2 & 3 & 0 & 1 \\ -2 & 1 & 4 & -1 \\ -5 & 1 & 1 & 5 \end{pmatrix}$$

Example

$$\det\begin{pmatrix} 1 & -1 & -1 & -1 \\ -2 & 3 & 0 & 1 \\ -2 & 1 & 4 & -1 \\ -5 & 1 & 1 & 5 \end{pmatrix} = \det\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & 0 & 3 & -2 \\ -4 & 0 & 0 & 4 \end{pmatrix}$$

Example

$$
\begin{aligned}
\det\begin{pmatrix} 1 & -1 & -1 & -1 \\ -2 & 3 & 0 & 1 \\ -2 & 1 & 4 & -1 \\ -5 & 1 & 1 & 5 \end{pmatrix} &= \det\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & 0 & 3 & -2 \\ -4 & 0 & 0 & 4 \end{pmatrix} \\
&= \det\begin{pmatrix} -2 & -1 & -1 & -1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 4 \end{pmatrix}
\end{aligned}
$$

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$$\det\begin{pmatrix} 1 & -1 & -1 & -1 \\ -2 & 3 & 0 & 1 \\ -2 & 1 & 4 & -1 \\ -5 & 1 & 1 & 5 \end{pmatrix} = \det\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & 0 & 3 & -2 \\ -4 & 0 & 0 & 4 \end{pmatrix}$$

$$= \det\begin{pmatrix} -2 & -1 & -1 & -1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

$$= -2 \cdot 2 \cdot 3 \cdot 4$$

$$= -48.$$