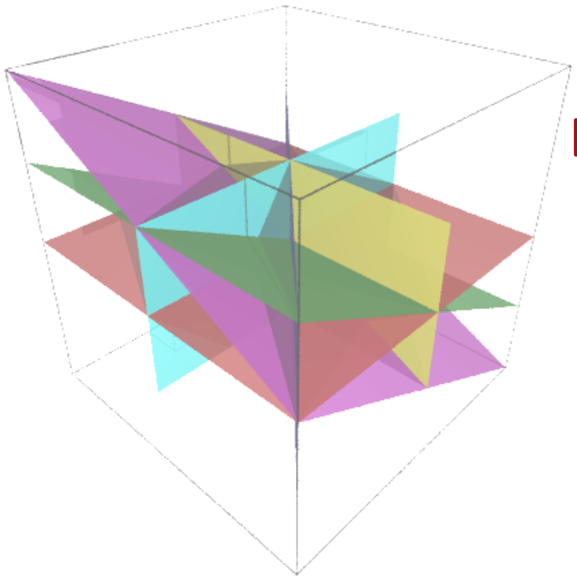

Math 201

Section F03

October 29, 2021

Existence and uniqueness of the determinant

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Define $d \colon M_{n \times n}(F) \to F$ recursively by

$$d(A) := \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \, d(A^{1j}) \tag{1}$$

for $n > 1$ and $d(A) = a$ if $A = [a]$ is a 1×1 matrix.

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Exercise. The function d is multilinear, alternating, and normalized.

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Uniqueness. The value of any multilinear, alternating, and normalized function of the rows of a matrix is completely determined (via any choice for row reduction). So there is only one multilinear, alternating, normalized function.

Laplace expansion

Let $A \in M_{n\times n}(F)$, and fix any $k \in \{1, \ldots, n\}$. Then

$$\det(A) = \sum_{j=1}^{n} (-1)^{k+j} A_{kj} \det(A^{kj}).$$

Generalized Laplace expansion

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$A^{IJ} =$ the $k \times k$ submatrix of A formed by the intersection of rows indexed by I and the columns indexed by J

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Then

$$\det(A) = \sum_J (-1)^{|I|+|J|} \det(A^{IJ}) \det(\bar{A}^{IJ})$$

where the sum is over all k-element subsets J of $\{1, \ldots, n\}$.