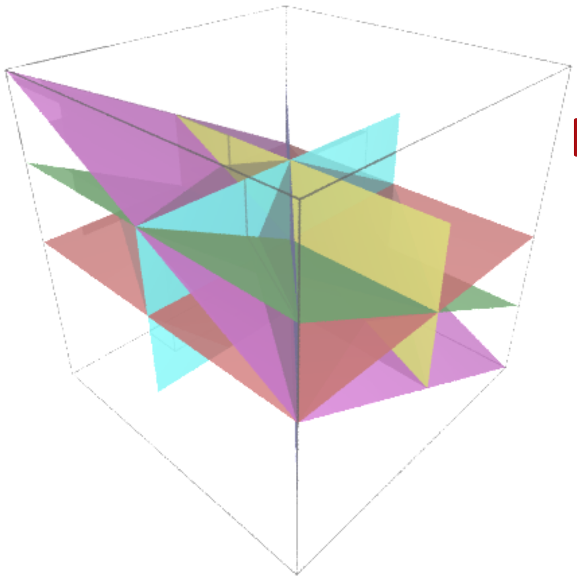

Math 201

Section F03

October 13, 2021

Review + some nice notation

$$\begin{aligned} f\colon \mathbb{R}[x]_{\leq 2} &\to \mathbb{R}[x]_{\leq 3} \\ p &\mapsto xp + 2p' \end{aligned}$$

Find the matrix representing f with respect to the ordered bases $\alpha = \langle 1, x, x^2 \rangle$ for $\mathbb{R}[x]_{\leq 2}$ and $\beta = \langle 1, x, x^2, x^3 \rangle$.

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$$f(1) = x, \quad f(x) = x^2 + 2, \quad f(x^2) = x^3 + 4x.$$

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The matrix is then

$$\begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \end{array} \begin{array}{c} \begin{array}{ccc} f(1) & f(x) & f(x^2) \end{array} \\ \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Change of basis

$\alpha = \langle v_1, \ldots, v_n \rangle$ ordered basis for V

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If $V = F^n$, then $\phi_\alpha \colon F^n \to F^n$ and $v_j \in F^n$ for all j.

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If $V = F^n$, then $\phi_\alpha \colon F^n \to F^n$ and $v_j \in F^n$ for all j.

Note: In any case, $\phi_\alpha(v_j) = e_j$.

Matrix representing coordinate mapping

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$$Pe_j = v_j$$

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Thus, $P^{-1}(v_j) = e_j = \phi_\alpha(v_j)$ for all v in the basis α.

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for all j.

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for all j. So $M = P^{-1}$. $\square$

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$$\begin{array}{ccc} F^n & \xrightarrow{L_A} & F^m \\ \phi_\alpha \downarrow \wr & & \wr \downarrow \phi_\beta \\ F^n & \xrightarrow{[L_A]_\alpha^\beta} & F^m \end{array}$$

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The mapping L_B is what L_A becomes after changing bases to α for the domain and β for the codomain.

Change of basis

Proposition.
Let $A \in M_{m\times n}(F)$, and consider the linear mapping $L_A \colon F^n \to F^m$ determined by A, i.e., $L(v) = Av$ for each $v \in F^n$.

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Let P be the $n \times n$ matrix with j-th column v_j for $j = 1, \ldots, n$, and let Q be the $m \times m$ matrix with j-th column w_j for $j = 1, \ldots, m$.

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Then the matrix B representing L_A with respect to the bases α and β is $B = Q^{-1}AP$:

$$\begin{array}{ccc} F^n & \xrightarrow{\ A\ } & F^m \\ {\scriptstyle P^{-1}}\big\downarrow \wr & & \wr \big\downarrow {\scriptstyle Q^{-1}} \\ F^n & \xrightarrow{\ B\ } & F^m. \end{array}$$

Example

$$\begin{aligned} f\colon \mathbb{Q}^3 &\to \mathbb{Q}^2 \\ (x, y, z) &\mapsto (x + 3y + 2z, 2y + z), \end{aligned}$$

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Bases:

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$$A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \end{pmatrix} \quad P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

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$$B = Q^{-1}AP = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & -2 & -3 \\ 1 & 4 & 6 \end{pmatrix}.$$

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Conjugation

Definition Let $A, B \in M_{n\times n}(F)$. Then A is *similar* to B, denoted $A \sim B$ if there exists an invertible matrix $P \in M_{n\times n}(F)$ such that $B = P^{-1}AP$.

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Definition Let $A, B \in M_{n\times n}(F)$. Then A is *similar* to B, denoted $A \sim B$ if there exists an invertible matrix $P \in M_{n\times n}(F)$ such that $B = P^{-1}AP$.

Proposition. Similarity is an equivalence relation on $M_{m\times n}(F)$.

Example

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad \alpha = \langle (1,1,1), (1,0,-1), (0,1,-1) \rangle$$

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$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix}.$$

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$$B = P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Example

$$A^2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix},$$

$$A^4 = \begin{pmatrix} 6 & 5 & 5 \\ 5 & 6 & 5 \\ 5 & 5 & 6 \end{pmatrix}, \quad A^5 = \begin{pmatrix} 10 & 11 & 11 \\ 11 & 10 & 11 \\ 11 & 11 & 10 \end{pmatrix}$$

$$A^6 = \begin{pmatrix} 22 & 21 & 21 \\ 21 & 22 & 21 \\ 21 & 21 & 22 \end{pmatrix}, \quad A^7 = \begin{pmatrix} 42 & 43 & 43 \\ 43 & 42 & 43 \\ 43 & 43 & 42 \end{pmatrix}$$

Example

$$
\begin{aligned}
A^k &= PB^kP^{-1} \\
&= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}^k \left(\frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix} \right) \\
&= \frac{1}{3} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 2^k & 0 & 0 \\ 0 & (-1)^k & 0 \\ 0 & 0 & (-1)^k \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix} \\
&= \frac{1}{3} \begin{pmatrix} 2^k + 2\,(-1)^k & 2^k - (-1)^k & 2^k - (-1)^k \\ 2^k - (-1)^k & 2^k + 2\,(-1)^k & 2^k - (-1)^k \\ 2^k - (-1)^k & 2^k - (-1)^k & 2^k + 2\,(-1)^k \end{pmatrix}
\end{aligned}
$$