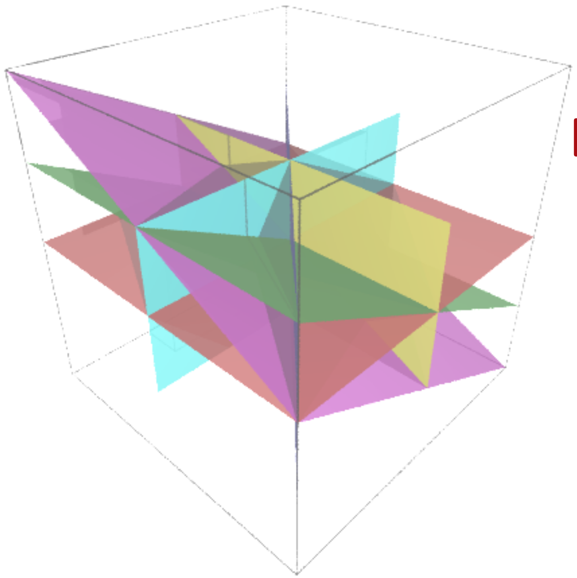

Math 201

Section F03

October 11, 2021

Review

Definition. If A is an $m \times p$ matrix and B is a $p \times n$ matrix, then AB is the $m \times n$ matrix with (i, j)-th entry

$$(AB)_{ij} := \sum_{k=1}^{p} A_{ik} B_{kj}.$$

Review

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Proposition. Let A be an $m \times n$ matrix, B an $n \times r$ matrix, both over a field F, and $\lambda \in F$.

1. $\lambda(AB) = (\lambda A)B = A(\lambda B)$.
2. $A(BC) = (AB)C$ for all $r \times s$ matrices C.
3. $A(B + C) = AB + AC$ for all $n \times r$ matrices C.
4. $(C + D)A = CA + DA$ for all $r \times m$ matrices C and D.

Diagonal matrices

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Example.

$$\mathrm{diag}(1, 4, 0, 6) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}.$$

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We have $AI_n = A$ and $I_nB = B$ whenever these products make sense.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}.$$

Inverses

If $AB = I_n$, then A is a *left inverse* for B and B is a *right inverse* for A.

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$$AB = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

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But,

$$BA = \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \neq I_3.$$

Matrix inverses

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The following are equivalent:

(i) A is invertible.

(ii) $\mathrm{rank}(A) = n$.

(iii) The reduced echelon form of A is I_n.

Matrix inversion algorithm

Problem: determine whether the following real matrix has an inverse, and if it does, calculate it:

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Equivalently, see if there are real numbers $a, \ldots, i$ such that

$$\begin{pmatrix} 0 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

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Equivalent to three calculations:

$$\begin{pmatrix} 0 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} a \\ d \\ g \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} b \\ e \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

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Matrix inversion algorithm

Corresponding augmented matrices:

$$\left(\begin{array}{rrr|r} 0 & 3 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right), \quad \left(\begin{array}{rrr|r} 0 & 3 & -1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{array}\right), \quad \left(\begin{array}{rrr|r} 0 & 3 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right).$$

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Solve simultaneously:

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Back to the original systems of equations, we get:

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Therefore,

$$\begin{pmatrix} 0 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1/4 & 1/4 & 3/4 \\ 1/4 & 1/4 & -1/4 \\ -1/4 & 3/4 & -3/4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

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In particular, A has a right inverse if and only if $\mathrm{rank}(A) = n$.

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Therefore, if $A, B \in M_{n\times n}$ and $AB = I_n$, then $BA = I_n$.

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