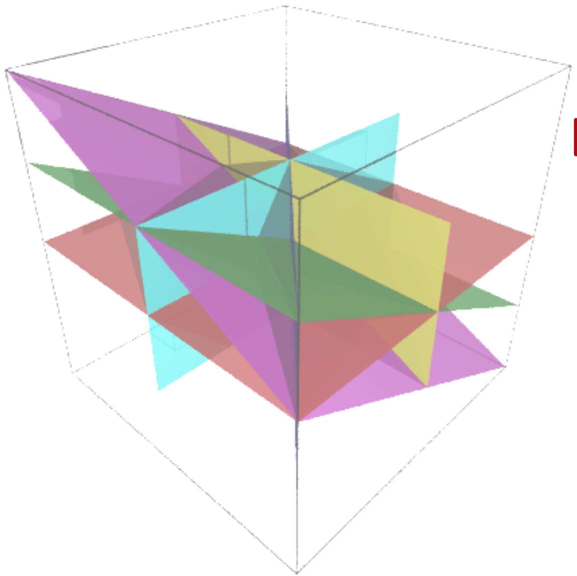

Math 201

Section F03

October 15, 2021

Review of change of basis stuff

$f \colon V \to W$

Review of change of basis stuff

$$f\colon V \to W \qquad \alpha = \langle v_1, \ldots, v_n \rangle, \quad \beta = \langle w_1, \ldots, w_n \rangle$$

Review of change of basis stuff

$$f\colon V \to W \qquad \alpha = \langle v_1, \ldots, v_n \rangle, \quad \beta = \langle w_1, \ldots, w_n \rangle$$

$$\begin{array}{ccc} V & \xrightarrow{\;f\;} & W \\ {\scriptstyle \phi_\alpha}\downarrow\wr & & \wr\downarrow{\scriptstyle \phi_\beta} \\ F^n & \xrightarrow{\;[f]_\alpha^\beta\;} & F^m \end{array}$$

Review of change of basis stuff

$$f\colon V \to W \qquad \alpha = \langle v_1, \dots, v_n \rangle, \quad \beta = \langle w_1, \dots, w_n \rangle$$

$$\begin{array}{ccc} V & \xrightarrow{\ f\ } & W \\ {\scriptstyle \phi_\alpha}\downarrow\wr & & \wr\downarrow{\scriptstyle \phi_\beta} \\ F^n & \xrightarrow{[f]_\alpha^\beta} & F^m \end{array}$$

$$\phi_\alpha\colon V \xrightarrow{\sim} F^n$$

$$v = a_1 v_1 + \cdots + a_n v_n \mapsto (a_1, \dots, a_n)$$

$$\phi_\beta\colon W \xrightarrow{\sim} F^m$$

$$w = b_1 w_1 + \cdots + b_n w_m \mapsto (b_1, \dots, b_m)$$

Review of change of basis stuff

$$L_A\colon F^n \to F^m, \qquad \alpha = \langle v_1, \dots, v_n \rangle, \quad \beta = \langle w_1, \dots, w_n \rangle$$

Review of change of basis stuff

$$L_A \colon F^n \to F^m, \qquad \alpha = \langle v_1, \dots, v_n \rangle, \quad \beta = \langle w_1, \dots, w_n \rangle$$

$$\begin{array}{ccc} F^n & \xrightarrow{A} & F^m \\ {\scriptstyle P^{-1}}\downarrow\wr & & \wr\downarrow{\scriptstyle Q^{-1}} \\ F^n & \xrightarrow{B} & F^m \end{array}$$

Review of change of basis stuff

$$L_A\colon F^n \to F^m, \qquad \alpha = \langle v_1, \ldots, v_n \rangle, \quad \beta = \langle w_1, \ldots, w_n \rangle$$

$$\begin{array}{ccc} F^n & \xrightarrow{A} & F^m \\ {\scriptstyle P^{-1}}\downarrow\wr & & \wr\downarrow{\scriptstyle Q^{-1}} \\ F^n & \xrightarrow{B} & F^m \end{array}$$

$$P = \begin{pmatrix} | & \cdots & | \\ v_1 & \ddots & v_n \\ | & \cdots & | \end{pmatrix} \qquad Q = \begin{pmatrix} | & \cdots & | \\ w_1 & \ddots & w_m \\ | & \cdots & | \end{pmatrix}$$

Review of change of basis stuff

$$L_A\colon F^n \to F^m, \qquad \alpha = \langle v_1, \ldots, v_n \rangle, \quad \beta = \langle w_1, \ldots, w_n \rangle$$

$$\begin{array}{ccc} F^n & \xrightarrow{A} & F^m \\ {\scriptstyle P^{-1}}\downarrow\wr & & \wr\downarrow{\scriptstyle Q^{-1}} \\ F^n & \xrightarrow{B} & F^m \end{array}$$

$$P = \begin{pmatrix} | & \cdots & | \\ v_1 & \ddots & v_n \\ | & \cdots & | \end{pmatrix} \qquad Q = \begin{pmatrix} | & \cdots & | \\ w_1 & \ddots & w_m \\ | & \cdots & | \end{pmatrix}$$

$$\boxed{B = Q^{-1}AP}$$

Review of change of basis stuff

$$L_A \colon F^n \to F^n, \qquad \alpha = \langle v_1, \dots, v_n \rangle$$

Review of change of basis stuff

$$L_A \colon F^n \to F^n, \qquad \alpha = \langle v_1, \dots, v_n \rangle$$

$$\begin{array}{ccc} F^n & \xrightarrow{A} & F^n \\ {\scriptstyle P^{-1}}\big\downarrow\wr & & \wr\big\downarrow{\scriptstyle P^{-1}} \\ F^n & \xrightarrow{B} & F^n \end{array}$$

Review of change of basis stuff

$$L_A \colon F^n \to F^n, \qquad \alpha = \langle v_1, \ldots, v_n \rangle$$

$$\begin{array}{ccc} F^n & \xrightarrow{A} & F^n \\ {\scriptstyle P^{-1}}\downarrow\wr & & \wr\downarrow{\scriptstyle P^{-1}} \\ F^n & \xrightarrow{B} & F^n \end{array}$$

$$\boxed{B = P^{-1}AP}$$

$$P = \begin{pmatrix} | & \cdots & | \\ v_1 & \ddots & v_n \\ | & \cdots & | \end{pmatrix}$$

Example

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

What matrix represents the linear function $L_A \colon F^3 \to F^3$ with respect to the ordered basis $\alpha = \langle (1,1,1), (1,0,-1), (0,1,-1) \rangle$?

Example

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

What matrix represents the linear function $L_A \colon F^3 \to F^3$ with respect to the ordered basis $\alpha = \langle (1,1,1), (1,0,-1), (0,1,-1) \rangle$?

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

Example

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

What matrix represents the linear function $L_A \colon F^3 \to F^3$ with respect to the ordered basis $\alpha = \langle (1,1,1), (1,0,-1), (0,1,-1) \rangle$?

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}, \quad B = P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Example

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

What matrix represents the linear function $L_A \colon F^3 \to F^3$ with respect to the ordered basis $\alpha = \langle (1,1,1), (1,0,-1), (0,1,-1) \rangle$?

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}, \quad B = P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

For instance,

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Example

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

What matrix represents the linear function $L_A \colon F^3 \to F^3$ with respect to the ordered basis $\alpha = \langle (1,1,1), (1,0,-1), (0,1,-1) \rangle$?

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}, \quad B = P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

For instance,

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad Av = \lambda v$$

HW problem

If A and B are $n \times n$ matrices, prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

HW problem

If A and B are $n \times n$ matrices, prove that $\text{tr}(AB) = \text{tr}(BA)$.

Solution.

$$\text{tr}(AB) = \sum_{i=1}^{n} (AB)_{ii}$$

HW problem

If A and B are $n \times n$ matrices, prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

Solution.

$$\begin{aligned} \mathrm{tr}(AB) &= \sum_{i=1}^{n} (AB)_{ii} \\ &= \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} B_{ki} \end{aligned}$$

HW problem

If A and B are $n \times n$ matrices, prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

Solution.

$$\begin{aligned}
\mathrm{tr}(AB) &= \sum_{i=1}^{n} (AB)_{ii} \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} B_{ki} \\
&= \sum_{k=1}^{n} \sum_{i=1}^{n} A_{ik} B_{ki}
\end{aligned}$$

HW problem

If A and B are $n \times n$ matrices, prove that $\text{tr}(AB) = \text{tr}(BA)$.

Solution.

$$\begin{aligned}
\text{tr}(AB) &= \sum_{i=1}^{n} (AB)_{ii} \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} B_{ki} \\
&= \sum_{k=1}^{n} \sum_{i=1}^{n} A_{ik} B_{ki} \\
&= \sum_{k=1}^{n} \sum_{i=1}^{n} B_{ki} A_{ik}
\end{aligned}$$

HW problem

If A and B are $n \times n$ matrices, prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

Solution.

$$
\begin{aligned}
\mathrm{tr}(AB) &= \sum_{i=1}^{n} (AB)_{ii} \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} B_{ki} \\
&= \sum_{k=1}^{n} \sum_{i=1}^{n} A_{ik} B_{ki} \\
&= \sum_{k=1}^{n} \sum_{i=1}^{n} B_{ki} A_{ik} \\
&= \sum_{k=1}^{n} (BA)_{kk}
\end{aligned}
$$

HW problem

If A and B are $n \times n$ matrices, prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Solution.

$$\begin{aligned}
\operatorname{tr}(AB) &= \sum_{i=1}^{n} (AB)_{ii} \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} B_{ki} \\
&= \sum_{k=1}^{n} \sum_{i=1}^{n} A_{ik} B_{ki} \\
&= \sum_{k=1}^{n} \sum_{i=1}^{n} B_{ki} A_{ik} \\
&= \sum_{k=1}^{n} (BA)_{kk} \\
&= \operatorname{tr}(BA).
\end{aligned}$$

HW problem

Therefore,

$$\mathrm{tr}(P^{-1}AP)$$

HW problem

Therefore,

$$\operatorname{tr}(P^{-1}AP) = \operatorname{tr}((P^{-1}A)P)$$

HW problem

Therefore,

$$\text{tr}(P^{-1}AP) = \text{tr}((P^{-1}A)P) = \text{tr}(P(P^{-1}A))$$

HW problem

Therefore,

$$\begin{aligned} \operatorname{tr}(P^{-1}AP) &= \operatorname{tr}((P^{-1}A)P) = \operatorname{tr}(P(P^{-1}A)) \\ &= \operatorname{tr}((PP^{-1})A) \end{aligned}$$

HW problem

Therefore,

$$\begin{aligned}\operatorname{tr}(P^{-1}AP) &= \operatorname{tr}((P^{-1}A)P) = \operatorname{tr}(P(P^{-1}A)) \\ &= \operatorname{tr}((PP^{-1})A) = \operatorname{tr}(I_nA)\end{aligned}$$

HW problem

Therefore,

$$\begin{aligned}\operatorname{tr}(P^{-1}AP) &= \operatorname{tr}((P^{-1}A)P) = \operatorname{tr}(P(P^{-1}A)) \\ &= \operatorname{tr}((PP^{-1})A) = \operatorname{tr}(I_nA) \\ &= \operatorname{tr}(A).\end{aligned}$$

HW problem

Therefore,

$$\begin{aligned}\operatorname{tr}(P^{-1}AP) &= \operatorname{tr}((P^{-1}A)P) = \operatorname{tr}(P(P^{-1}A)) \\ &= \operatorname{tr}((PP^{-1})A) = \operatorname{tr}(I_n A) \\ &= \operatorname{tr}(A).\end{aligned}$$

Later: $\operatorname{tr}(A) =$ sum of eigenvalues of A

Determinants

Definition. The *determinant*, det: $M_{n\times n}(F) \to F$ is a multilinear, alternating function of the rows of square matrix, normalized so that its value on the identity matrix is 1.

Determinants

Definition. The *determinant*, $\det\colon M_{n\times n}(F) \to F$ is a multilinear, alternating function of the rows of square matrix, normalized so that its value on the identity matrix is 1.

1. *Multilinear.* The determinant is a linear function with respect to each row:

$$\begin{aligned}\det(r_1, \dots, r_{i-1}, \lambda\, r_i + r_i', r_{i+1}, \dots, r_n) &= \lambda \det(r_1, \dots, r_{i-1}, r_i, r_{i+1}, \dots, r_n) \\ &\quad + \det(r_1, \dots, r_{i-1}, r_i', r_{i+1}, \dots, r_n).\end{aligned}$$

Determinants

Definition. The *determinant*, $\det\colon M_{n\times n}(F) \to F$ is a multilinear, alternating function of the rows of square matrix, normalized so that its value on the identity matrix is 1.

1. *Multilinear.* The determinant is a linear function with respect to each row:

$$\begin{aligned}\det(r_1, \ldots, r_{i-1}, \lambda\, r_i + r_i', r_{i+1}, \ldots, r_n) &= \lambda \det(r_1, \ldots, r_{i-1}, r_i, r_{i+1}, \ldots, r_n)\\ &\quad + \det(r_1, \ldots, r_{i-1}, r_i', r_{i+1}, \ldots, r_n).\end{aligned}$$

2. *Alternating.* The determinant is zero if two of its arguments are equal:

$$\det(r_1, \ldots, r_n) = 0$$

if $r_i = r_j$ for some $i \neq j$.

Determinants

Definition. The *determinant*, $\det\colon M_{n\times n}(F) \to F$ is a multilinear, alternating function of the rows of square matrix, normalized so that its value on the identity matrix is 1.

1. *Multilinear.* The determinant is a linear function with respect to each row:

$$\begin{aligned}\det(r_1, \ldots, r_{i-1}, \lambda\, r_i + r_i', r_{i+1}, \ldots, r_n) &= \lambda \det(r_1, \ldots, r_{i-1}, r_i, r_{i+1}, \ldots, r_n) \\ &\quad + \det(r_1, \ldots, r_{i-1}, r_i', r_{i+1}, \ldots, r_n).\end{aligned}$$

2. *Alternating.* The determinant is zero if two of its arguments are equal:

$$\det(r_1, \ldots, r_n) = 0$$

   if $r_i = r_j$ for some $i \neq j$.
3. *Normalized.* $\det(I_n) = \det(e_1, \ldots, e_n) = 1$.

Main theorem

Theorem. For each $n \geq 0$, there exists a unique determinant function.

Secret importance of determinants

The determinant of $A \in M_{n \times n}(\mathbb{R})$ is $\pm$ the volume of the parallelepiped spanned by the rows (or columns) of A:

$$P(A) = \left\{ p \in \mathbb{R}^n \colon p = \sum_{i=1}^{n} \lambda_i r_i \text{ where } 0 \le \lambda_i \le 1 \text{ for all } i \right\}.$$

Secret importance of determinants

The determinant of $A \in M_{n\times n}(\mathbb{R})$ is $\pm$ the volume of the parallelepiped spanned by the rows (or columns) of A:

$$P(A) = \left\{ p \in \mathbb{R}^n \colon p = \sum_{i=1}^{n} \lambda_i r_i \text{ where } 0 \leq \lambda_i \leq 1 \text{ for all } i \right\}.$$

The sign defines the *orientation* of the ordered list of rows of A.

Secret importance of determinants

The determinant of $A \in M_{n \times n}(\mathbb{R})$ is $\pm$ the volume of the parallelepiped spanned by the rows (or columns) of A:

$$P(A) = \left\{ p \in \mathbb{R}^n \colon p = \sum_{i=1}^{n} \lambda_i r_i \text{ where } 0 \leq \lambda_i \leq 1 \text{ for all } i \right\}.$$

The sign defines the *orientation* of the ordered list of rows of A. Multivariable calculus builds on this to compute volumes of more general subsets of $\mathbb{R}^n$.

Determinants & row operations

Proposition. Let $A, B \in M_{n\times n}(F)$.

Determinants & row operations

Proposition. Let $A, B \in M_{n\times n}(F)$.

1. If B is obtained from A by swapping two rows, then $\det(B) = -\det(A)$.

Determinants & row operations

Proposition. Let $A, B \in M_{n\times n}(F)$.

1. If B is obtained from A by swapping two rows, then $\det(B) = -\det(A)$.
2. If B is obtained from A by scaling a row by a scalar λ, then $\det(B) = \lambda \det(A)$ (even if $\lambda = 0$).

Determinants & row operations

Proposition. Let $A, B \in M_{n\times n}(F)$.

1. If B is obtained from A by swapping two rows, then $\det(B) = -\det(A)$.
2. If B is obtained from A by scaling a row by a scalar λ, then $\det(B) = \lambda \det(A)$ (even if $\lambda = 0$).
3. If B is obtained from A by adding a scalar multiple of one row to another row, then $\det(B) = \det(A)$.

Determinants & row operations

Proposition. Let $A, B \in M_{n\times n}(F)$.

1. If B is obtained from A by swapping two rows, then $\det(B) = -\det(A)$.
2. If B is obtained from A by scaling a row by a scalar λ, then $\det(B) = \lambda \det(A)$ (even if $\lambda = 0$).
3. If B is obtained from A by adding a scalar multiple of one row to another row, then $\det(B) = \det(A)$.

Corollary. Let $A \in M_{n\times n}(F)$, and let E be the reduced row echelon form of A. Then there exists a non-zero $k \in F$ such that $\det(A) = k \det(E)$.

Example

Compute the determinant of a general 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Example

Compute

$$\det \begin{pmatrix} 1 & 2 & -2 \\ 9 & 4 & 0 \\ 2 & 2 & 4 \end{pmatrix}.$$

Example

$$\det \begin{pmatrix} 4 & 2 & -3 & 8 \\ 0 & 5 & 1 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Example

$$\det \begin{pmatrix} 4 & 2 & -3 & 8 \\ 0 & 5 & 1 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \end{pmatrix} = (4 \cdot 5 \cdot 2 \cdot 3) \det \begin{pmatrix} 1 & 1/2 & -3/2 & 4 \\ 0 & 1 & 1/5 & 3/5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Example

$$\det \begin{pmatrix} 4 & 2 & -3 & 8 \\ 0 & 5 & 1 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \end{pmatrix} = (4 \cdot 5 \cdot 2 \cdot 3) \det \begin{pmatrix} 1 & 1/2 & -3/2 & 4 \\ 0 & 1 & 1/5 & 3/5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= (4 \cdot 5 \cdot 2 \cdot 3) \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Example

$$\det\begin{pmatrix} 4 & 2 & -3 & 8 \\ 0 & 5 & 1 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \end{pmatrix} = (4\cdot 5\cdot 2\cdot 3)\det\begin{pmatrix} 1 & 1/2 & -3/2 & 4 \\ 0 & 1 & 1/5 & 3/5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= (4\cdot 5\cdot 2\cdot 3)\det\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= (4\cdot 5\cdot 2\cdot 3)\cdot 1 = 120.$$

Example

Compute

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Example

Compute

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\det(r_1, r_2, r_3, \vec{0}) = \det(r_1, r_2, r_3, 0 \cdot \vec{0})$$

Example

Compute

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} \det(r_1, r_2, r_3, \vec{0}) &= \det(r_1, r_2, r_3, 0 \cdot \vec{0}) \\ &= 0 \cdot \det(r_1, r_2, r_3, \vec{0}) \end{aligned}$$

Example

Compute

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} \det(r_1, r_2, r_3, \vec{0}) &= \det(r_1, r_2, r_3, 0 \cdot \vec{0}) \\ &= 0 \cdot \det(r_1, r_2, r_3, \vec{0}) \\ &= 0 \end{aligned}$$

Example

Compute

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} \det(r_1, r_2, r_3, \vec{0}) &= \det(r_1, r_2, r_3, 0 \cdot \vec{0}) \\ &= 0 \cdot \det(r_1, r_2, r_3, \vec{0}) \\ &= 0 \end{aligned}$$

In general: if A has a row of zeros, then $\det(A) = 0$.

Upper-triangular matrices

Definition. A matrix $A \in M_{n \times n}(F)$ is *upper-triangular* if $A_{ij} = 0$ for all $i > j$.

Upper-triangular matrices

Definition. A matrix $A \in M_{n\times n}(F)$ is *upper-triangular* if $A_{ij} = 0$ for all $i > j$.

Examples.

1. $$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

Upper-triangular matrices

Definition. A matrix $A \in M_{n \times n}(F)$ is *upper-triangular* if $A_{ij} = 0$ for all $i > j$.

Examples.

1.
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

2.
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

Upper-triangular matrices

Definition. A matrix $A \in M_{n \times n}(F)$ is *upper-triangular* if $A_{ij} = 0$ for all $i > j$.

Examples.

1.
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

2.
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

3. Any square matrix in reduced row echelon form.

Upper-triangular matrices

Proposition. The determinant of an upper-triangular matrix is the product of its diagonal elements.

Upper-triangular matrices

Proposition. The determinant of an upper-triangular matrix is the product of its diagonal elements.

Proof. See today's lecture notes.

Invertibility criterion

Proposition. Let $A \in M_{n\times n}(F)$. The following are equivalent:

1. $\det(A) \neq 0$,
2. $\mathrm{rank}(A) = n$,
3. A is invertible, i.e., A has an inverse.

Invertibility criterion

Proposition. Let $A \in M_{n\times n}(F)$. The following are equivalent:

1. $\det(A) \neq 0$,
2. $\mathrm{rank}(A) = n$,
3. A is invertible, i.e., A has an inverse.

Proof sketch.

- $\det(A) = k \det(E)$ with $k \neq 0$.

Invertibility criterion

Proposition. Let $A \in M_{n\times n}(F)$. The following are equivalent:

1. $\det(A) \neq 0$,
2. $\mathrm{rank}(A) = n$,
3. A is invertible, i.e., A has an inverse.

Proof sketch.

- $\det(A) = k \det(E)$ with $k \neq 0$.
- $\det(E) \neq 0 \quad \Leftrightarrow \quad E = I_n$