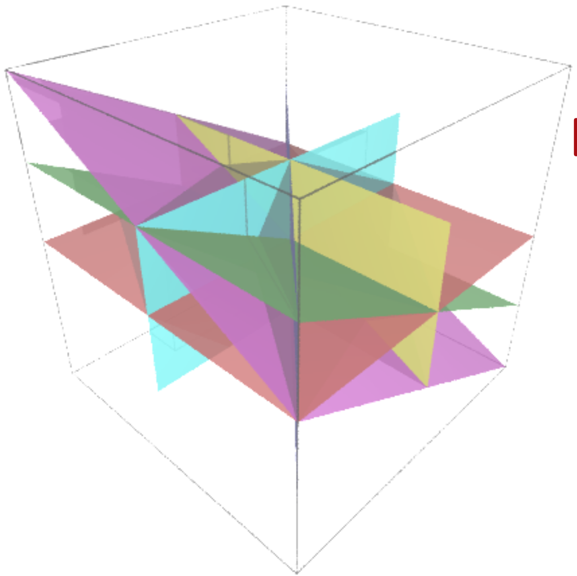


Math 201

Section F03

October 6, 2021

Review

There is a one-to-one correspondence between matrices $A \in M_{m\times n}(F)$ and linear functions $L\colon F^n \to F^m$.

Review

There is a one-to-one correspondence between matrices $A \in M_{m\times n}(F)$ and linear functions $L\colon F^n \to F^m$.

($\Rightarrow$) Given $A \in M_{m\times n}(F)$, we get

$$\begin{aligned} L_A\colon F^n &\to F^m \\ x &\mapsto Ax. \end{aligned}$$

Review

There is a one-to-one correspondence between matrices $A \in M_{m\times n}(F)$ and linear functions $L\colon F^n \to F^m$.

($\Rightarrow$) Given $A \in M_{m\times n}(F)$, we get

$$\begin{aligned} L_A\colon F^n &\to F^m \\ x &\mapsto Ax. \end{aligned}$$

($\Leftarrow$) Given $L\colon F^n \to F^m$, we get $A_L \in M_{m\times n}$ having j-th column equal to $L(e_j)$.

Review

There is a one-to-one correspondence between matrices $A \in M_{m \times n}(F)$ and linear functions $L\colon F^n \to F^m$.

($\Rightarrow$) Given $A \in M_{m \times n}(F)$, we get

$$\begin{aligned} L_A\colon F^n &\to F^m \\ x &\mapsto Ax. \end{aligned}$$

($\Leftarrow$) Given $L\colon F^n \to F^m$, we get $A_L \in M_{m \times n}$ having j-th column equal to $L(e_j)$.

Example.

$$\begin{aligned} L\colon F^3 &\to F^2 \\ (x, y, z) &\mapsto (2x + 3y + z, 4x + 7y + 5z) \end{aligned}$$

Review

There is a one-to-one correspondence between matrices $A \in M_{m\times n}(F)$ and linear functions $L\colon F^n \to F^m$.

($\Rightarrow$) Given $A \in M_{m\times n}(F)$, we get

$$\begin{aligned} L_A\colon F^n &\to F^m \\ x &\mapsto Ax. \end{aligned}$$

($\Leftarrow$) Given $L\colon F^n \to F^m$, we get $A_L \in M_{m\times n}$ having j-th column equal to $L(e_j)$.

Example.

$$\begin{aligned} L\colon F^3 &\to F^2 \\ (x,y,z) &\mapsto (2x+3y+z, 4x+7y+5z) \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 2 & 3 & 1 \\ 4 & 7 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Review

There is a one-to-one correspondence between
matrices $A \in M_{m \times n}(F)$ and linear functions $L\colon F^n \to F^m$.

($\Rightarrow$) Given $A \in M_{m \times n}(F)$, we get

$$\begin{aligned} L_A\colon F^n &\to F^m \\ x &\mapsto Ax. \end{aligned}$$

($\Leftarrow$) Given $L\colon F^n \to F^m$, we get $A_L \in M_{m \times n}$ having j-th column equal to $L(e_j)$.

Example.

$$\begin{aligned} L\colon F^3 &\to F^2 \\ (x, y, z) &\mapsto (2x + 3y + z, 4x + 7y + 5z) \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 2 & 3 & 1 \\ 4 & 7 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + 3y + z \\ 4x + 7y + 5z \end{pmatrix}$$

Review

$f: V \to W$

Review

$f \colon V \to W$

V with ordered basis $\mathcal{B} = \langle v_1, \ldots, v_n \rangle$
W with ordered basis $\mathcal{D} = \langle w_1, \ldots, w_n \rangle$

Review

$f\colon V \to W$

V with ordered basis $\mathcal{B} = \langle v_1, \ldots, v_n \rangle$
W with ordered basis $\mathcal{D} = \langle w_1, \ldots, w_n \rangle$

$$\begin{array}{ccc} V & \xrightarrow{\;f\;} & W \\ {\scriptstyle \phi_{\mathcal{B}}}\big\downarrow\wr & & \wr\big\downarrow{\scriptstyle \phi_{\mathcal{D}}} \\ F^n & \xrightarrow{\;L\;} & F^m \end{array}$$

Review

$f \colon V \to W$

V with ordered basis $\mathcal{B} = \langle v_1, \ldots, v_n \rangle$
W with ordered basis $\mathcal{D} = \langle w_1, \ldots, w_n \rangle$

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & W \\ {\scriptstyle \phi_{\mathcal{B}}} \downarrow \wr & & \wr \downarrow {\scriptstyle \phi_{\mathcal{D}}} \\ F^n & \xrightarrow{\quad L \quad} & F^m \end{array}$$

$$L := \phi_{\mathcal{D}} \circ f \circ \phi_{\mathcal{B}}^{-1}$$

Review

$f\colon V \to W$

V with ordered basis $\mathcal{B} = \langle v_1, \ldots, v_n \rangle$
W with ordered basis $\mathcal{D} = \langle w_1, \ldots, w_n \rangle$

$$\begin{array}{ccc} V & \xrightarrow{\;f\;} & W \\ {\scriptstyle \phi_{\mathcal{B}}}\big\downarrow{\scriptstyle \wr} & & {\scriptstyle \wr}\big\downarrow{\scriptstyle \phi_{\mathcal{D}}} \\ F^n & \xrightarrow{\;L\;} & F^m \end{array}$$

$$L := \phi_{\mathcal{D}} \circ f \circ \phi_{\mathcal{B}}^{-1} \quad \rightsquigarrow \text{ matrix } [f]_{\mathcal{B}}^{\mathcal{D}}$$

Review

$f\colon V \to W$

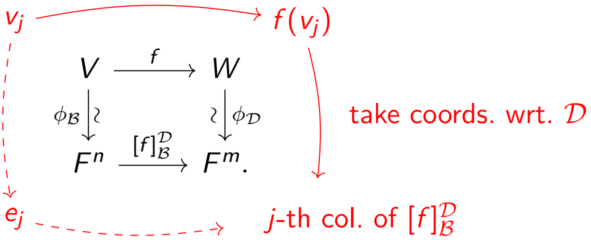
V with ordered basis $\mathcal{B} = \langle v_1, \ldots, v_n \rangle$
W with ordered basis $\mathcal{D} = \langle w_1, \ldots, w_n \rangle$

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle \phi_{\mathcal{B}}}\downarrow\wr & & \wr\downarrow{\scriptstyle \phi_{\mathcal{D}}} \\ F^n & \xrightarrow{L} & F^m \end{array}$$

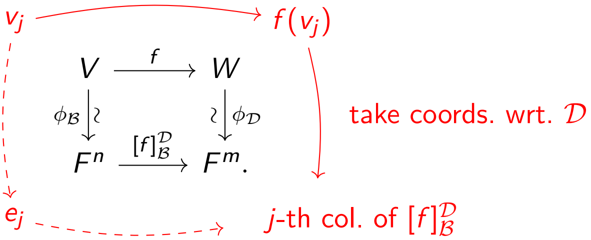
$$L := \phi_{\mathcal{D}} \circ f \circ \phi_{\mathcal{B}}^{-1} \quad \rightsquigarrow \text{ matrix } [f]_{\mathcal{B}}^{\mathcal{D}}$$

$$[f]_{\mathcal{B}}^{\mathcal{D}}[v]_{\mathcal{B}} = [f(v)]_{\mathcal{D}}.$$

Review

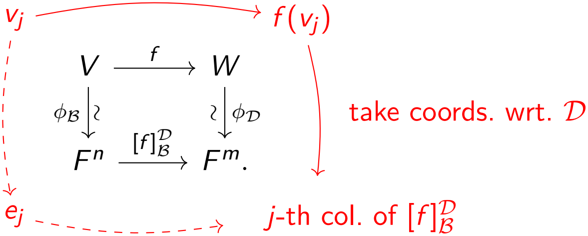

v_j
$f(v_j)$
$V \xrightarrow{f} W$
$\phi_{\mathcal{B}}$
$\phi_{\mathcal{D}}$
$F^n \xrightarrow{[f]_{\mathcal{B}}^{\mathcal{D}}} F^m.$
take coords. wrt. $\mathcal{D}$
e_j
j-th col. of $[f]_{\mathcal{B}}^{\mathcal{D}}$

Review



The j-th column of $[f]_{\mathcal{B}}^{\mathcal{D}}$ gives the coordinates of $f(e_j)$ with respect to $\mathcal{D}$.

Review



The j-th column of $[f]_{\mathcal{B}}^{\mathcal{D}}$ gives the coordinates of $f(e_j)$ with respect to $\mathcal{D}$.

Question. What happens in the special case where $V = F^n$, $W = F^m$ and both $\mathcal{B}$ and $\mathcal{D}$ are the standard bases?

Ranks

Proposition. Let V and W be finite-dimensional vector spaces with ordered bases $\mathcal{B}$ and $\mathcal{D}$, respectively. Let $f\colon V \to W$ be a linear transformation. Then

$$\mathrm{rank}(f) = \mathrm{rank}([f]_{\mathcal{B}}^{\mathcal{D}}).$$

Ranks

Proposition. Let V and W be finite-dimensional vector spaces with ordered bases $\mathcal{B}$ and $\mathcal{D}$, respectively. Let $f\colon V \to W$ be a linear transformation. Then

$$\operatorname{rank}(f) = \operatorname{rank}([f]_{\mathcal{B}}^{\mathcal{D}}).$$

Proof. First consider the case $f\colon F^n \to F^m$, and we choose the standard bases for domain and codomain. Then $f(x) = Ax$ for some matrix $A \in M_{m\times n}$.

Ranks

$$Ax = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} := \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}$$

$$= (a_{11}x_1 + a_{12}x_2 \cdots + a_{1n}x_n,$$
$$a_{21}x_1 + a_{22}x_2 \cdots + a_{2n}x_n, \dots, a_{m1}x_1 + a_{m2}x_2 \cdots + a_{mn}x_n).$$

Ranks

$$Ax = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} := \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}$$

$$= (a_{11}x_1 + a_{12}x_2 \cdots + a_{1n}x_n,$$
$$a_{21}x_1 + a_{22}x_2 \cdots + a_{2n}x_n, \dots, a_{m1}x_1 + a_{m2}x_2 \cdots + a_{mn}x_n).$$

$$= x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}.$$

Ranks

Proposition. Let V and W be finite-dimensional vector spaces with ordered bases $\mathcal{B}$ and $\mathcal{D}$, respectively. Let $f\colon V \to W$ be a linear transformation. Then

$$\mathrm{rank}(f) = \mathrm{rank}([f]_{\mathcal{B}}^{\mathcal{D}}).$$

Ranks

Proposition. Let V and W be finite-dimensional vector spaces with ordered bases $\mathcal{B}$ and $\mathcal{D}$, respectively. Let $f\colon V \to W$ be a linear transformation. Then

$$\mathrm{rank}(f) = \mathrm{rank}([f]_{\mathcal{B}}^{\mathcal{D}}).$$

Proof. General case:

Ranks

Proposition. Let V and W be finite-dimensional vector spaces with ordered bases $\mathcal{B}$ and $\mathcal{D}$, respectively. Let $f\colon V \to W$ be a linear transformation. Then

$$\mathrm{rank}(f) = \mathrm{rank}([f]_{\mathcal{B}}^{\mathcal{D}}).$$

Proof. General case:

$$\begin{array}{ccc} V & \xrightarrow{\;f\;} & W \\ {\scriptstyle \phi_{\mathcal{B}}}\downarrow\wr & & \wr\downarrow{\scriptstyle \phi_{\mathcal{D}}} \\ F^n & \xrightarrow{\;L_A\;} & F^m \end{array}$$

Ranks

Proposition. Let V and W be finite-dimensional vector spaces with ordered bases $\mathcal{B}$ and $\mathcal{D}$, respectively. Let $f \colon V \to W$ be a linear transformation. Then

$$\operatorname{rank}(f) = \operatorname{rank}([f]_{\mathcal{B}}^{\mathcal{D}}).$$

Proof. General case:

$$\begin{array}{ccc} V & \xrightarrow{\;f\;} & W \\ {\scriptstyle \phi_{\mathcal{B}}}\downarrow\wr & & \wr\downarrow{\scriptstyle \phi_{\mathcal{D}}} \\ F^n & \xrightarrow{\;L_A\;} & F^m \end{array}$$

where $A = [f]_{\mathcal{B}}^{\mathcal{D}}$.

Ranks

Proposition. Let V and W be finite-dimensional vector spaces with ordered bases $\mathcal{B}$ and $\mathcal{D}$, respectively. Let $f\colon V \to W$ be a linear transformation. Then

$$\operatorname{rank}(f) = \operatorname{rank}([f]_{\mathcal{B}}^{\mathcal{D}}).$$

Proof. General case:

$$\begin{array}{ccc} V & \xrightarrow{\;f\;} & W \\ {\scriptstyle \phi_{\mathcal{B}}}\downarrow{\scriptstyle \wr} & & {\scriptstyle \wr}\downarrow{\scriptstyle \phi_{\mathcal{D}}} \\ F^n & \xrightarrow{\;L_A\;} & F^m \end{array}$$

where $A = [f]_{\mathcal{B}}^{\mathcal{D}}$. Then

$$\operatorname{rank}(A) = \operatorname{rank}(L_A)$$

Ranks

Proposition. Let V and W be finite-dimensional vector spaces with ordered bases $\mathcal{B}$ and $\mathcal{D}$, respectively. Let $f\colon V \to W$ be a linear transformation. Then

$$\mathrm{rank}(f) = \mathrm{rank}([f]_{\mathcal{B}}^{\mathcal{D}}).$$

Proof. General case:

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & W \\ {\scriptstyle \phi_{\mathcal{B}}}\downarrow{\scriptstyle \wr} & & {\scriptstyle \wr}\downarrow{\scriptstyle \phi_{\mathcal{D}}} \\ F^n & \xrightarrow{\quad L_A \quad} & F^m \end{array}$$

where $A = [f]_{\mathcal{B}}^{\mathcal{D}}$. Then

$$\mathrm{rank}(A) = \mathrm{rank}(L_A) := \dim(\mathrm{im}(L_A))$$

Ranks

Proposition. Let V and W be finite-dimensional vector spaces with ordered bases $\mathcal{B}$ and $\mathcal{D}$, respectively. Let $f: V \to W$ be a linear transformation. Then

$$\operatorname{rank}(f) = \operatorname{rank}([f]_{\mathcal{B}}^{\mathcal{D}}).$$

Proof. General case:

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle \phi_{\mathcal{B}}}\downarrow\wr & & \wr\downarrow{\scriptstyle \phi_{\mathcal{D}}} \\ F^n & \xrightarrow{L_A} & F^m \end{array}$$

where $A = [f]_{\mathcal{B}}^{\mathcal{D}}$. Then

$$\operatorname{rank}(A) = \operatorname{rank}(L_A) := \dim(\operatorname{im}(L_A)) = \dim(\operatorname{im}(L_A \circ \phi_{\mathcal{B}}))$$

Ranks

Proposition. Let V and W be finite-dimensional vector spaces with ordered bases $\mathcal{B}$ and $\mathcal{D}$, respectively. Let $f\colon V \to W$ be a linear transformation. Then

$$\operatorname{rank}(f) = \operatorname{rank}([f]_{\mathcal{B}}^{\mathcal{D}}).$$

Proof. General case:

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & W \\ {\scriptstyle \phi_{\mathcal{B}}}\big\downarrow \wr & & \wr \big\downarrow {\scriptstyle \phi_{\mathcal{D}}} \\ F^n & \xrightarrow{\quad L_A \quad} & F^m \end{array}$$

where $A = [f]_{\mathcal{B}}^{\mathcal{D}}$. Then

$$\begin{aligned} \operatorname{rank}(A) = \operatorname{rank}(L_A) &:= \dim(\operatorname{im}(L_A)) = \dim(\operatorname{im}(L_A \circ \phi_{\mathcal{B}})) \\ &= \dim(\operatorname{im}(\phi_{\mathcal{D}} \circ f)) \end{aligned}$$

Ranks

Proposition. Let V and W be finite-dimensional vector spaces with ordered bases $\mathcal{B}$ and $\mathcal{D}$, respectively. Let $f\colon V \to W$ be a linear transformation. Then

$$\mathrm{rank}(f) = \mathrm{rank}([f]_{\mathcal{B}}^{\mathcal{D}}).$$

Proof. General case:

$$\begin{CD} V @>{f}>> W \\ @V{\phi_{\mathcal{B}}}V{\wr}V @V{\wr}V{\phi_{\mathcal{D}}}V \\ F^n @>{L_A}>> F^m \end{CD}$$

where $A = [f]_{\mathcal{B}}^{\mathcal{D}}$. Then

$$\mathrm{rank}(A) = \mathrm{rank}(L_A) := \dim(\mathrm{im}(L_A)) = \dim(\mathrm{im}(L_A \circ \phi_{\mathcal{B}}))$$
$$= \dim(\mathrm{im}(\phi_{\mathcal{D}} \circ f)) = \dim(\mathrm{im}(f))$$

Ranks

Proposition. Let V and W be finite-dimensional vector spaces with ordered bases $\mathcal{B}$ and $\mathcal{D}$, respectively. Let $f\colon V \to W$ be a linear transformation. Then

$$\operatorname{rank}(f) = \operatorname{rank}([f]_{\mathcal{B}}^{\mathcal{D}}).$$

Proof. General case:

$$\begin{array}{ccc} V & \xrightarrow{\ f\ } & W \\ {\scriptstyle \phi_{\mathcal{B}}}\downarrow\wr & & \wr\downarrow{\scriptstyle \phi_{\mathcal{D}}} \\ F^n & \xrightarrow{\ L_A\ } & F^m \end{array}$$

where $A = [f]_{\mathcal{B}}^{\mathcal{D}}$. Then

$$\operatorname{rank}(A) = \operatorname{rank}(L_A) := \dim(\operatorname{im}(L_A)) = \dim(\operatorname{im}(L_A \circ \phi_{\mathcal{B}}))$$
$$= \dim(\operatorname{im}(\phi_{\mathcal{D}} \circ f)) = \dim(\operatorname{im}(f)) =: \operatorname{rank}(f). \qquad \square$$

Ranks

Proposition. Let V and W be finite-dimensional vector spaces with ordered bases $\mathcal{B}$ and $\mathcal{D}$, respectively. Let $f\colon V \to W$ be a linear transformation. Then

$$\operatorname{rank}(f) = \operatorname{rank}([f]_{\mathcal{B}}^{\mathcal{D}}).$$

Ranks

Proposition. Let V and W be finite-dimensional vector spaces with ordered bases $\mathcal{B}$ and $\mathcal{D}$, respectively. Let $f\colon V \to W$ be a linear transformation. Then

$$\operatorname{rank}(f) = \operatorname{rank}([f]_{\mathcal{B}}^{\mathcal{D}}).$$

Corollary. With notation as above, let $A = [f]_{\mathcal{B}}^{\mathcal{D}} \in M_{m\times n}(F)$.

(a) f is surjective if and only if $\operatorname{rank}(A) = m = \dim(W)$.

(b) f is injective if and only if $\operatorname{rank}(A) = n = \dim(V)$.

(c) f is an isomorphism if and only if $\operatorname{rank}(A) = m = n$.

Composition of linear functions

$$f \colon \mathbb{R}^4 \to \mathbb{R}^2$$
$$(x, y, z, w) \mapsto (2x - z + 3w, x - y + 4z)$$

and

$$g \colon \mathbb{R}^2 \to \mathbb{R}^3$$
$$(s, t) \mapsto (5s - t, 2t, -3s).$$

Compute $g \circ f$.

Composition of linear functions

$$g \circ f \colon V \xrightarrow{f} W \xrightarrow{g} U.$$

Composition of linear functions

$$g \circ f \colon V \xrightarrow{f} W \xrightarrow{g} U.$$

$$\begin{aligned} V: &\quad \mathcal{B} = \langle v_1, \ldots, v_n \rangle \\ W: &\quad \mathcal{C} = \langle w_1, \ldots, w_\ell \rangle \\ U: &\quad \mathcal{D} = \langle u_1, \ldots, u_m \rangle \end{aligned}$$

Composition of linear functions

$$g \circ f \colon V \xrightarrow{f} W \xrightarrow{g} U.$$

$$V : \quad \mathcal{B} = \langle v_1, \ldots, v_n \rangle$$
$$W : \quad \mathcal{C} = \langle w_1, \ldots, w_\ell \rangle$$
$$U : \quad \mathcal{D} = \langle u_1, \ldots, u_m \rangle$$

$$P := [g]_{\mathcal{C}}^{\mathcal{D}} \quad \text{and} \quad Q = [f]_{\mathcal{B}}^{\mathcal{C}}.$$

Composition of linear functions

$$g \circ f \colon V \xrightarrow{f} W \xrightarrow{g} U.$$

$$\begin{aligned} V : &\quad \mathcal{B} = \langle v_1, \ldots, v_n \rangle \\ W : &\quad \mathcal{C} = \langle w_1, \ldots, w_\ell \rangle \\ U : &\quad \mathcal{D} = \langle u_1, \ldots, u_m \rangle \end{aligned}$$

$$P := [g]_{\mathcal{C}}^{\mathcal{D}} \quad \text{and} \quad Q = [f]_{\mathcal{B}}^{\mathcal{C}}.$$

$$\begin{array}{ccccc} V & \xrightarrow{\quad f \quad} & W & \xrightarrow{\quad g \quad} & U \\ {\scriptstyle \phi_{\mathcal{B}}} \downarrow \wr & & \wr \downarrow {\scriptstyle \phi_{\mathcal{C}}} & & \wr \downarrow {\scriptstyle \phi_{\mathcal{D}}} \\ F^n & \xrightarrow{\quad Q \quad} & F^\ell & \xrightarrow{\quad P \quad} & F^m \end{array}$$

Matrix multiplication

Definition. Let $P \in M_{m\times\ell}(F)$ and $Q \in M_{\ell\times n}(F)$, then the product $PQ \in M_{m\times n}(F)$ is defined by

$$(PQ)_{ij} = \sum_{k=1}^{\ell} P_{ik} Q_{kj}.$$

Example.

$$\begin{pmatrix} 5 & -1 \\ 0 & 2 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 & 3 \\ 1 & -1 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 9 & 1 & -9 & 15 \\ 2 & -2 & 8 & 0 \\ -6 & 0 & 3 & -9 \end{pmatrix}.$$