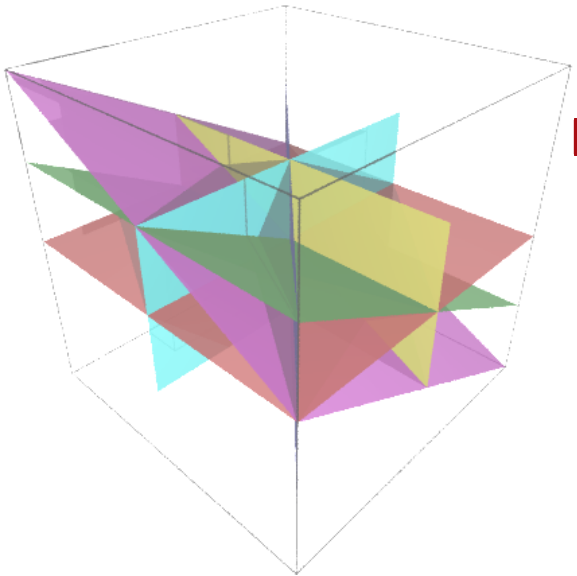

Math 201

Section F03

October 4, 2021

Goal

Goal: encode linear transformations in matrices

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Example.

$$\begin{aligned} L\colon F^3 &\to F^2 \\ (x, y, z) &\mapsto (2x + 3y + z, 4x + 7y + 5z) \end{aligned}$$

Goal

Goal: encode linear transformations in matrices

Example.

$$L\colon F^3 \to F^2$$
$$(x, y, z) \mapsto (2x + 3y + z, 4x + 7y + 5z)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 2 & 3 & 1 \\ 4 & 7 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

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Goal: encode linear transformations in matrices

Example.

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$$(x, y, z) \mapsto (2x + 3y + z, 4x + 7y + 5z)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 2 & 3 & 1 \\ 4 & 7 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + 3y + z \\ 4x + 7y + 5z \end{pmatrix}$$

Dot product

Dot product: $\cdot\colon F^n \times F^n \to F$

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$u = (a_1, \ldots, a_n), v = (b_1, \ldots, b_n) \in F^n$

$$u \cdot v := \sum_{i=1}^{n} a_i b_i = a_1 b_1 + \cdots + a_n b_n.$$

Matrix $\to$ linear mapping

Definition. Linear mapping associated with $A \in M_{m\times n}(F)$:

$$\begin{aligned} L_A \colon F^n &\to F^n \\ x &\mapsto Ax \end{aligned}$$

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where

$$Ax = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} := \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}$$

$$\begin{aligned} = (&a_{11}x_1 + a_{12}x_2 \cdots + a_{1n}x_n, \\ &a_{21}x_1 + a_{22}x_2 \cdots + a_{2n}x_n, \dots, a_{m1}x_1 + a_{m2}x_2 \cdots + a_{mn}x_n). \end{aligned}$$

Matrix → linear mapping

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$$= (a_{11}x_1 + a_{12}x_2 \cdots + a_{1n}x_n,$$

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Matrix $\rightarrow$ linear mapping

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$$a_{21}x_1 + a_{22}x_2 \cdots + a_{2n}x_n, \dots, a_{m1}x_1 + a_{m2}x_2 \cdots + a_{mn}x_n).$$

$$= x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}.$$

Linear mapping $\to$ matrix

Definition. The matrix associated with the linear function $L\colon F^n \to F^m$ is the element $A \in M_{m\times n}(F)$ whose j-th column is $L(e_j)$ where e_j is the j-th standard basis vector for F^n.

$$A = \begin{pmatrix} \vdots & \vdots & \ddots & \vdots \\ L(e_1) & L(e_2) & \dots & L(e_n) \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

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Example. What is the matrix associated with $L\colon F^2 \to F^3$ given by

$$L(x, y) = (3x + 4y, 6y, 4x + 2y)?$$

Matrix $\rightarrow$ linear mapping, general case

$f : V \rightarrow W$

Matrix $\to$ linear mapping, general case

$f\colon V \to W$

ordered basis for V: $\mathcal{B} = \langle v_1, \ldots, v_n \rangle$
ordered basis for W: $\mathcal{D} = \langle w_1, \ldots, w_m \rangle$

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Take coordinates:

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle \phi_{\mathcal{B}}}\big\downarrow \wr & & \wr \big\downarrow {\scriptstyle \phi_{\mathcal{D}}} \\ F^n & & F^m \end{array}$$

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