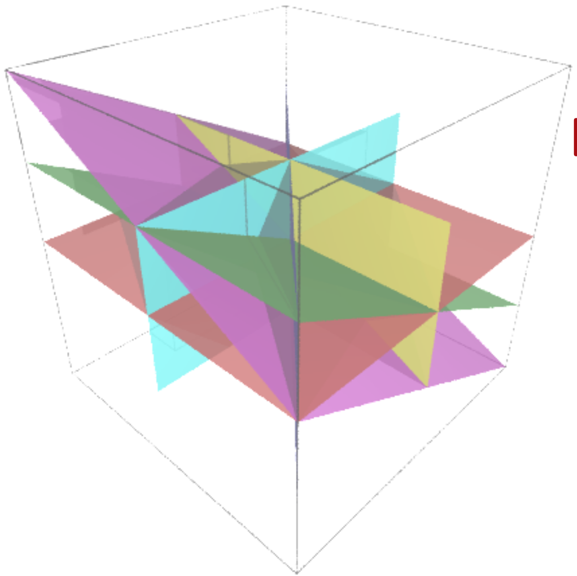

Math 201

Section F03

October 8, 2021

Composition of linear functions

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$$\begin{aligned} f\colon \mathbb{R}^4 &\to \mathbb{R}^2 \\ (x, y, z, w) &\mapsto (2x - z + 3w, x - y + 4z) \end{aligned}$$

and

$$\begin{aligned} g\colon \mathbb{R}^2 &\to \mathbb{R}^3 \\ (s, t) &\mapsto (5s - t, 2t, -3s). \end{aligned}$$

Compute $g \circ f$.

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Compute $g \circ f$.

$$\begin{aligned} (g \circ f) &= g(\underbrace{2x - z + 3w}_{s}, \underbrace{x - y + 4z}_{t}) \\ &= (5(2x - z + 3w) - (x - y + 4z), 2(x - y + 4z), -3(2x - z + 3w)) \\ &= (9x + y - 9z + 15w, 2x - 2y + 8z, -6x + 3z - 9w). \end{aligned}$$

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$$\begin{aligned} f(x,y,z,w) &= (2x - z + 3w, x - y + 4z) \\ g(s,t) &= (5s - t, 2t, -3s) \\ (g \circ f)(x,y,z,w) &= (9x + y - 9z + 15w, 2x - 2y + 8z, -6x + 3z - 9w) \end{aligned}$$

$$\begin{pmatrix} 5 & -1 \\ 0 & 2 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 & 3 \\ 1 & -1 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 9 & 1 & -9 & 15 \\ 2 & -2 & 8 & 0 \\ -6 & 0 & 3 & -9 \end{pmatrix}$$

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$$\begin{array}{ccccc} V & \xrightarrow{f} & W & \xrightarrow{g} & U \\ \phi_{\mathcal{B}} \downarrow \wr & & \wr \downarrow \phi_{\mathcal{C}} & & \wr \downarrow \phi_{\mathcal{D}} \\ F^n & \xrightarrow{Q} & F^\ell & \xrightarrow{P} & F^m \end{array}$$

Matrix multiplication

Definition. Let $P \in M_{m\times\ell}(F)$ and $Q \in M_{\ell\times n}(F)$, then the product $PQ \in M_{m\times n}(F)$ is defined by

$$(PQ)_{ij} = \sum_{k=1}^{\ell} P_{ik} Q_{kj}.$$

Example.

$$\begin{pmatrix} 5 & -1 \\ 0 & 2 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 & 3 \\ 1 & -1 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 9 & 1 & -9 & 15 \\ 2 & -2 & 8 & 0 \\ -6 & 0 & 3 & -9 \end{pmatrix}.$$

Composition of linear functions

Proposition. Let $f\colon V \to W$ and $g\colon W \to U$ be a linear functions, and fix ordered bases $\mathcal{B} = \langle v_1, \ldots, v_n \rangle$ for V, $\mathcal{C} = \langle w_1, \ldots, w_\ell \rangle$ for W, and $\mathcal{D} = \langle u_1, \ldots, u_m \rangle$ for U. Then we have

$$[g \circ f]_{\mathcal{B}}^{\mathcal{D}} = [g]_{\mathcal{C}}^{\mathcal{D}} [f]_{\mathcal{B}}^{\mathcal{C}}.$$

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V & \xrightarrow{\quad f \quad} & W & \xrightarrow{\quad g \quad} & U \\
\phi_{\mathcal{B}} \downarrow \wr & & \wr \downarrow \phi_{\mathcal{C}} & & \wr \downarrow \phi_{\mathcal{D}} \\
F^n & \xrightarrow{[f]_{\mathcal{B}}^{\mathcal{C}}} & F^\ell & \xrightarrow{[g]_{\mathcal{C}}^{\mathcal{D}}} & F^m
\end{array}$$

Matrix algebra

Proposition. Let A be an $m \times n$ matrix, B an $n \times r$ matrix, both over a field F, and $\lambda \in F$.

1. $\lambda(AB) = (\lambda A)B = A(\lambda B)$.
2. $A(BC) = (AB)C$ for all $r \times s$ matrices C.
3. $A(B + C) = AB + AC$ for all $n \times r$ matrices C.
4. $(C + D)A = CA + DA$ for all $r \times m$ matrices C and D.

Linear functions and matrices

Proposition. Let V and W be vectors spaces over F of dimension n and m, respectively. Then there is an isomorphism of vector spaces

$$\mathrm{Hom}(V, W) \to M_{m\times n}(F).$$