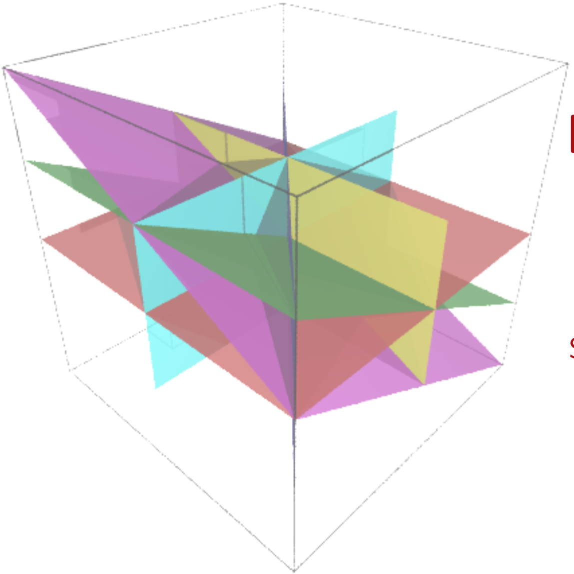

Math 201

Section F03

September 27, 2021

Linear transformations

Definition. Let V and W be vector spaces over a field F. A *linear transformation* from V to W is a function

$$f\colon V \to W$$

satisfying, for all $v, v' \in V$ and $\lambda \in F$,

$$f(v + v') = f(v) + f(v') \quad \text{and} \quad f(\lambda v) = \lambda f(v).$$

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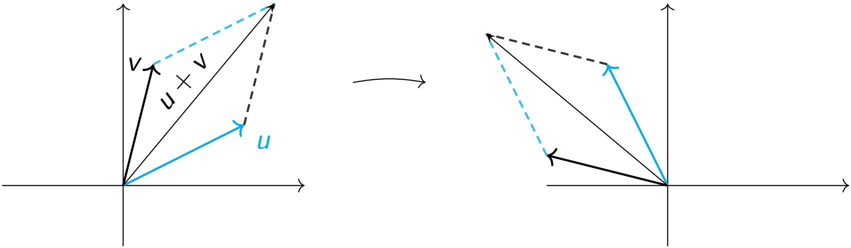
Example. Rotation about the origin in the plane $\mathbb{R}^2$ is a linear transformation:



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A linear function *preserves* addition and scalar multiplication:

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A linear subspace $U \subseteq V$ is *closed* under addition and scalar multiplication:

$$u, v \in U \quad \Rightarrow \quad u + \lambda v \in U.$$

Template for proving a mapping is linear

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$$\begin{aligned}
&f((x_1, y_1, z_1) + \lambda(x_2, y_2, z_2)) \\
&= f(x_1 + \lambda x_2, y_1 + \lambda y_2, z_1 + \lambda z_2) \\
&= (2(x_1 + \lambda x_2) + 3(y_1 + \lambda y_2), (x_1 + \lambda x_2) + (y_1 + \lambda y_2) - 3(z_1 + \lambda z_2)) \\
&= ((2x_1 + 3y_1) + \lambda(2x_2 + 3y_2), (x_1 + y_1 - 3z_1) + \lambda(x_2 + y_2 - 3z_2)) \\
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$$= f(x_1, y_1, z_1) + \lambda f(x_2, y_2, z_2).$$

Therefore, f preserves addition and scalar multiplication.

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Proposition. If $f\colon V \to W$ is linear, then

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Exercise. Is the following mapping linear?

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What about

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^2? \end{aligned}$$

A linear function is determined by its action on a basis

Example. Define a linear function $f : \mathbb{R}^2 \to M_{2\times 3}(\mathbb{R})$ by

$$f(1,0) = \begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 2 \end{pmatrix} \quad \text{and} \quad f(0,1) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}.$$

What is $f(2,-1)$?

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Example. Is there a linear function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ with $f(1,0) = (1,4)$ and $f(2,0) = (3,2)$?