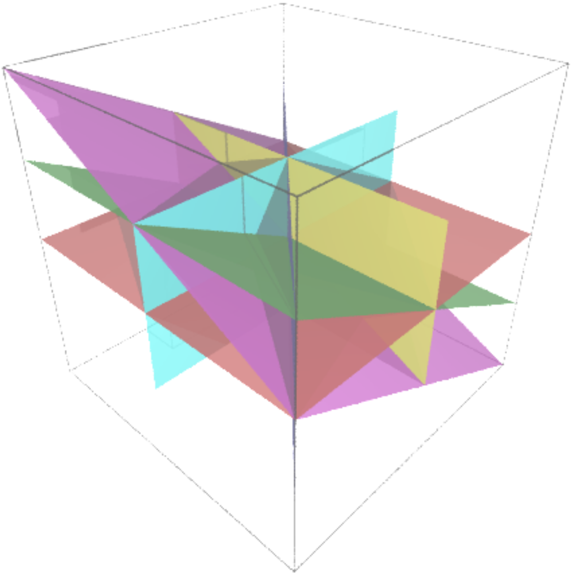

Math 201

Section F03

October 1, 2021

Review

Last time: $f\colon V \to W$

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Kernel $=$ stuff in V sent to 0_W

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$$\mathcal{R}(f) = \mathrm{im}(f) = f(V) = \{f(v) \in W : v \in V\}$$

$$\mathrm{rank}(f) = \dim(\mathrm{im}(f)).$$

Rank-nullity theorem

Theorem. (Rank-nullity theorem) Let $f\colon V \to W$ be a linear mapping, and suppose that V is finite-dimensional. Then

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Proof.

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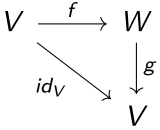

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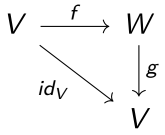

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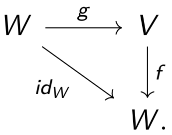

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$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ & \searrow^{id_V} & \downarrow g \\ & & V \end{array}$$

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The function g is called the *inverse of f* and denoted f^{-1}.

Examples of isomorphisms

$$f\colon M_{2\times 3} \to F^6$$
$$\begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix} \mapsto (u, v, w, x, y, z).$$

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$$\phi\colon F[x]_{\leq 3} \to F^4$$
$$a + bx + cx^2 + dx^3 \mapsto (a, b, c, d).$$

Isomophisms

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(bijective $=$ injective (1-1) and surjective (onto))

Isomophisms

Proposition 1. The linear mapping $f\colon V \to W$ is injective (i.e., one-to-one) if and only if $\ker(f) = \{0_V\}$.

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(a) The image of a dependent set is dependent.

(b) The image of an independent set is independent *provided* f is injective.

Isomorphisms

Proposition 3. A linear mapping $f\colon V \to W$ is an isomorphism if and only if $\ker(f) = \{0_V\}$ and $\operatorname{im}(f) = W$, (i.e., if and only if its kernel is trivial and it is surjective).

Isomorphisms

Theorem 4. Let V be a vector space over F. Then V is isomorphic to F^n if and only if $\dim V = n$.

Isomorphisms

Corollary 5. Let V and W be finite-dimensional vectors spaces. Then V and W are isomorphic if and only if they have the same dimension.

Isomorphisms

Proposition 6. Let $f\colon V \to W$ be a linear function, and let $\dim V = \dim W < \infty$. Then the following are equivalent:

(a) f is injective (1-1),

(b) f is surjective (onto),

(c) f is an isomorphism.

Weekend assignment

- Learn the statement and proof of the rank-nullity theorem.
- Learn the proof that $f\colon V \to W$ is injective if and only if $\ker(f) = \{0_v\}$.