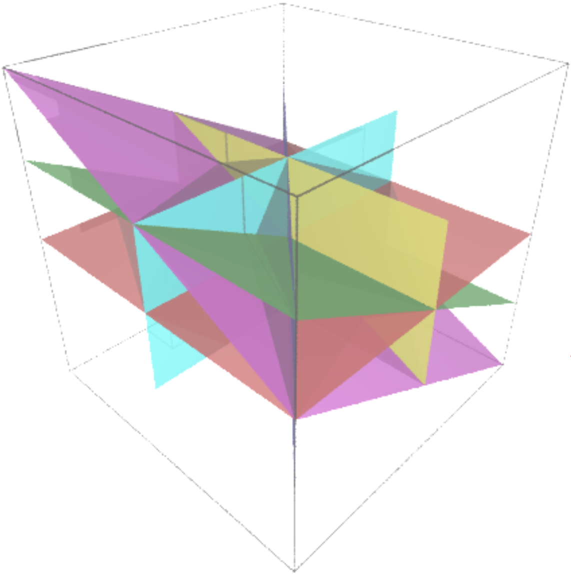

Math 201

Section F03

September 22, 2021

Dimension theorem (review)

Exchange Lemma. Suppose $B = \{v_1, \ldots, v_n\}$ is a basis for a vector space V over a field F. Further, suppose that

$$w = a_1 v_1 + \cdots + a_n v_n \in V \qquad (\star)$$

with $a_i \in F$, and such that $a_\ell \neq 0$ for some $\ell \in \{1, \ldots, n\}$. Let B' be the set of vectors obtained from B by exchanging w for v_ℓ, i.e., $B' := (B \setminus \{v_\ell\}) \cup \{w\}$. Then B' is also a basis for V.

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– Therefore, $B' = C$ and $n = |B| = |B'| = |C|$.

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– $\dim\{\vec{0}\} = 0$ (the basis is $\emptyset$, which has 0 elements).

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5. A basis is a minimal spanning set for V. (Here, "minimal" can mean the set has no strict subsets that also span V, or it can mean minimal in number of elements.)
6. A basis is a maximal linearly independent subset of V. (Here, "maximal" can mean there is no strict superset that is also linearly independent, or it can mean maximal in number.)

Game

$F := \mathbb{Z}/3\mathbb{Z}$. Points in F^4:

$$
\begin{array}{ccc}
(1,1,2,1) & (1,1,2,0) & (2,1,2,1) \\
(1,1,0,1) & (2,0,1,0) & (1,0,1,1) \\
(2,1,1,0) & (1,2,0,0) & (1,2,2,1) \\
(1,2,0,1) & (2,0,1,1) & (0,0,2,2)
\end{array}
$$

Goal: find subsets of size three of this array that sum to $(0,0,0,0)$.

Solutions.

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– $(1,1,2,1),(1,0,1,1),(1,2,0,1)$

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Game of Set