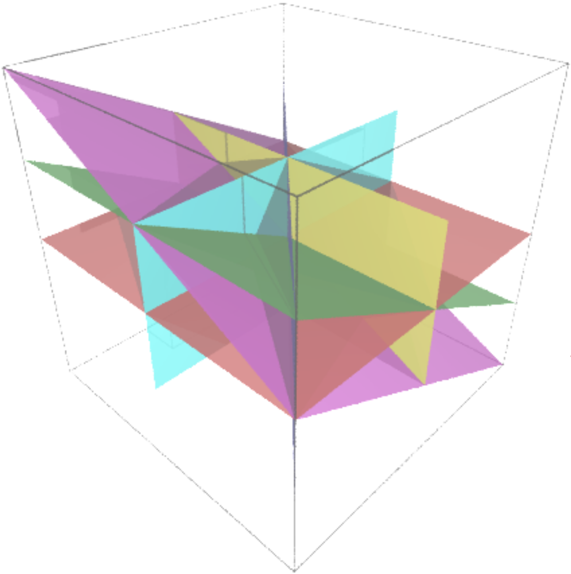

Math 201

Section F03

September 20, 2021

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- A set B is a *basis* for V if it
  - is linearly independent, and
  - spans V.
- If B is a basis for V, each element of V can be expressed uniquely as a linear combination of vectors in B.
- If $B = \langle v_1, \ldots, v_n \rangle$ is an ordered basis for V, then the *coordinates of $v \in V$ with respect to B* are $(a_1, \ldots, a_n)$ where

$$v = a_1 v_1 + \cdots + a_n v_n.$$

Exercise

Find the coordinates of $(7, -6) \in \mathbb{R}^2$ with respect to the ordered basis $B = \langle (5, 3), (1, 4) \rangle$.

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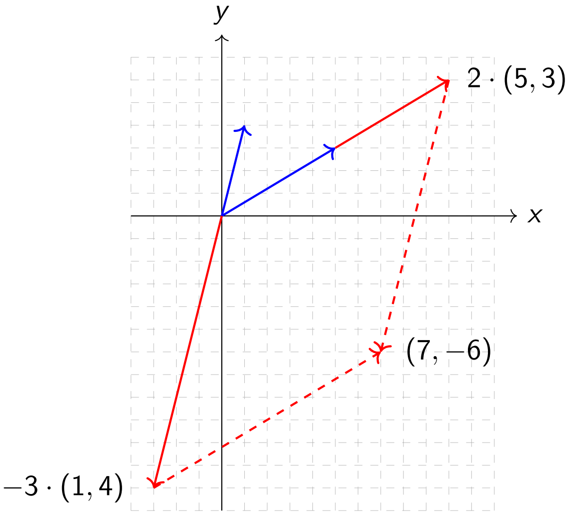

Figure 1: The coordinates of $(7, -6)$ with respect to the ordered b

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Definition. If V is a finite-dimensional vector space, then the *dimension* of V, denoted $\dim V$ or $\dim_F V$, if we want to make the scalar field explicit, is the number of elements in any of its bases.

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Then B' is also a basis for V.