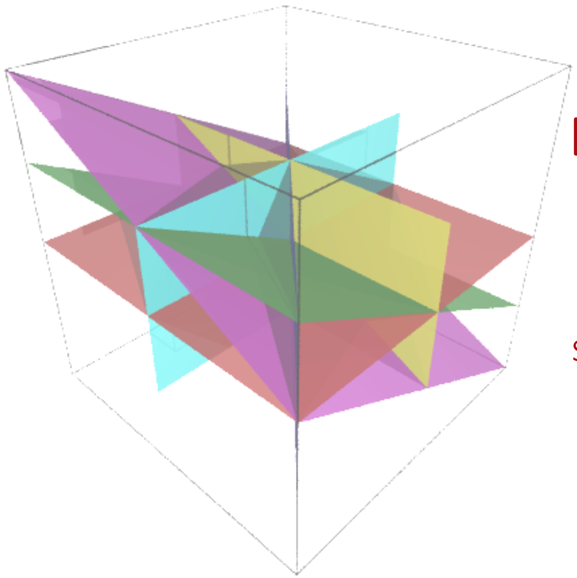

Math 201

Section F03

September 24, 2021

Row and column rank

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Example.

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 5 & 0 & 7 \\ 0 & 1 & 2 & 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 1 & 6 & 3 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Example

$$\operatorname{rowrank}\begin{pmatrix} 1 & 0 & 2 & 0 & 5 & 0 & 7 \\ 0 & 1 & 2 & 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 1 & 6 & 3 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = 3$$

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$$= (\mathbf{a}, \mathbf{b}, 2a + 2b, \mathbf{c}, 5a + 6c, 7b + 3c, 7a + b + 8c)$$

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$$\Longrightarrow a = b = c = 0.$$

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Conclusion: Row operations do not affect row rank.

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basis for $\mathrm{rowspace}(A)$: $\left\{ \left(1, 0, \frac{2}{3}, -4\right), \left(0, 1, -\frac{1}{3}, 4\right) \right\}$.

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– $\mathrm{rowrank}(A) = \mathrm{colrank}(A) =$ number of pivot columns of E.

Computing a basis for the column space

To prove: If $E_{j_1}, \ldots, E_{j_k}$ are the pivot columns E, then $A_{j_1}, \ldots, A_{j_k}$ are a basis for $\mathrm{colspace}(A)$.

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Note: the first two columns of E do not span $\mathrm{rowspace}(A)$.

Main technical result

$A \in M_{m\times n}(F)$ with reduce row echelon form E

columns of E: $E_1, \dots, E_n$

columns of A: $A_1, \dots, A_n$

Proposition. For $x_1, \dots, x_n \in F$,

$$x_1 A_1 + \cdots + x_n A_n = 0 \quad \Leftrightarrow \quad x_1 E_1 + \cdots + x_n E_n = 0.$$

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Proof. Write out $x_1 A_1 + \cdots + x_n A_n = 0$ longhand:

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = 0.$$

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Equivalently,

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ \vdots \qquad \vdots \qquad \vdots& \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

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Proof continued:

Row operations do not affect the solutions $(x_1, \ldots, x_n)$ to

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Corollary. If $E_{j_1}, \ldots, E_{j_k}$ are the pivot columns E, then $A_{j_1}, \ldots, A_{j_k}$ are a basis for $\mathrm{colspace}(A)$.

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Example.

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 5 & 0 & 7 \\ 0 & 1 & 2 & 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 1 & 6 & 3 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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Let A_ℓ be any column of A not listed above.

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Uniqueness of solutions

Consider the system

$$\begin{aligned} x_1 + 2x_2 + x_4 &= 3 \\ 3x_1 + 3x_2 + x_3 &= 1 \\ 7x_1 + 8x_2 + 2x_3 + 4x_4 &= 5 \end{aligned}$$

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$$A = \left(\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 3 \\ 3 & 3 & 1 & 0 & 1 \\ 7 & 8 & 2 & 4 & 5 \end{array}\right) \longrightarrow E = \left(\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & -4 & -\frac{7}{3} \\ 0 & 1 & -\frac{1}{3} & 4 & \frac{8}{3} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Uniqueness of solutions

$$\begin{aligned} a_{11}x_{11} + \cdots + a_{1n}x_n &= b_1 \\ \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_m + \cdots + a_{mn}x_n &= b_n \end{aligned}$$

has a unique solution if and only if it is consistent and $\mathrm{rank}(A) = n$ where

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}.$$

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$$
\begin{aligned}
a_{11}x_{11} + \cdots + a_{1n}x_n &= b_1 \\
\vdots \qquad \vdots \qquad \vdots \\
a_{m1}x_m + \cdots + a_{mn}x_n &= b_n
\end{aligned}
$$

has a unique solution if and only if it is consistent and $\mathrm{rank}(A) = n$ where

$$
A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}.
$$

If the system is homogeneous, there is a unique solution if and only if $\mathrm{rank}(A) = n$.