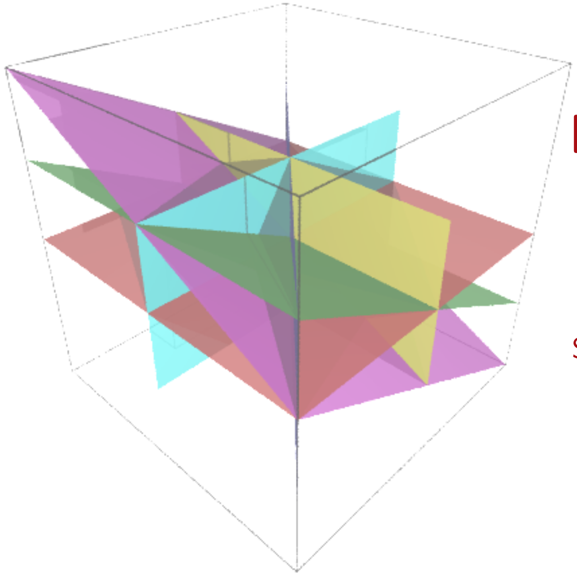

Math 201

Section F03

September 15, 2021

Linear dependence

Definition. A set $S \subset V$ is *linearly dependent* if there exist distinct[1] $u_1, \ldots, u_n \in S$, for some $n \geq 1$, and scalars $a_1, \ldots, a_n$, not all zero, such that

$$a_1 u_1 + \cdots + a_n u_n = 0.$$

[1] Note the easily forgotten but necessary word "distinct", here.

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We call the above expression a *non-trivial dependence relation* among the u_i.

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i.e., such that

$$(a_1 - a_2 - 5a_3, -a_1 + 3a_3, 2a_2 + 4a_3) = (0, 0, 0).$$

Linear dependence

Proposition 1. Let $S \subseteq V$. Then S is linearly dependent if and only if there exists $v \in S$ such that v is a linear combination of vectors in $S \setminus \{v\}$, i.e., if and only if $v \in \mathrm{Span}(S \setminus \{v\})$.

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Problem. Show $S = \{(1, -1, 0), (-1, 0, 2), (0, 1, 1)\} \subset \mathbb{R}$ is linearly independent.

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It follows that

$$\begin{aligned} a - \quad b \qquad &= 0 \\ -a \qquad + c &= 0 \\ 2b + c &= 0. \end{aligned}$$

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Applying Gaussian elemination:

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we see that the only solution is $a = b = c = 0$. $\square$

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2. Show that $a_1 = \cdots = a_n = 0$.

In general, do not assume $a_1 u_1 + \cdots + a_n u_n = 0$ with some $a_i \neq 0$ and derive a contradiction.

Linear independence

Problem. Show that $S = \{1 + x, 1 + x + x^2\} \subset P_2(\mathbb{R}) = \mathbb{R}[x]_{\leq 2}$ is linearly independent.

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Solution. Look for linear relations

$$c_1(2,0,0)+c_2(0,1,0)+c_3(2,2,0)+c_4(0,3,1)+c_5(3,0,1) = (0,0,0).$$

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Claim: $T = \{(2,0,0),(0,1,0),(0,3,1))$ is linearly independent and $\mathrm{Span}\, T = \mathrm{Span}(S)$.

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Note: The set T is a subset of the columns of M *not* of M'!

Coordinates

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In other words, if $v = \sum_{i=1}^{k} a_i u_i$ and $v = \sum_{i=1}^{\ell} b_i w_i$ for some nonzero $a_i, b_i \in F$ and some distinct $u_i \in S$ and distinct $w_i \in S$, then up to re-indexing, we have $k = \ell$, $u_i = w_i$, and $a_i = b_i$ for all i.