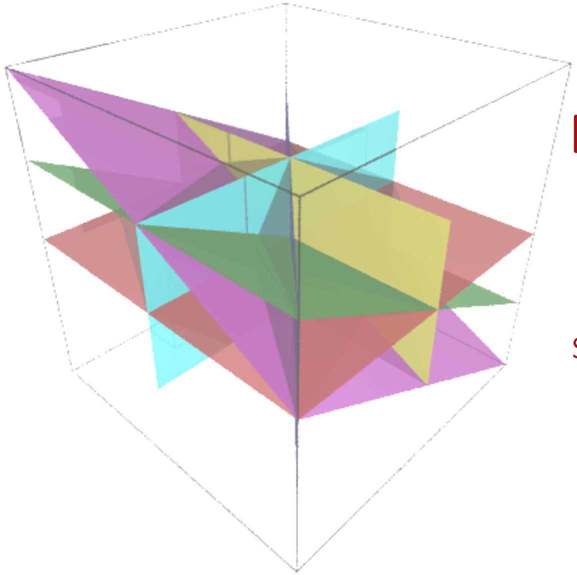

Math 201

Section F03

September 17, 2021

Bases

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Remarks.

- Our text uses "basis" to mean "ordered basis".
- It turns out that every vector space has a basis. (Although, *infinite dimensional* vector spaces may not have countable bases.

Existence of basis

Proved last time: Let $S \subseteq V$. Then S is linearly dependent if and only if there exists $v \in S$ such that v is a linear combination of vectors in $S \setminus \{v\}$, i.e., if and only if $v \in \mathrm{Span}(S \setminus \{v\})$.

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It follows that

Proposition. Any finite subset S of V has a linearly independent subset with the same span. In other words, if S is a finite set, then there is a subset of S that is a basis for $\mathrm{Span}(S)$.

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Since B is a basis, $\mathrm{Span}(B) = V$. $\square$

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$$W = \{(0,0,0),(2,1,0),(1,2,0),(2,0,1),(1,1,1),(0,2,1),$$
$$(1,0,2),(0,1,2),(2,2,2)\}$$

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The *coordinates of v with respect to the basis B* are the components of the vector $(a_1, \ldots, a_n) \in F^n$. We write

$$[v]_B = (a_1, \ldots, a_n).$$