MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE MONDAY WEEK 2

Remark 1. This is homework on the content covered Wednesday of Week 1. In general, homework on content delivered during course meeting n will be due at the start of course meeting $n+2$. You are encouraged to start working on the homework shortly after course meeting n in order to have time to ask questions during office hours or via our Slack channel.

Remark 2. Make sure to review the *Homework* portion of the syllabus before writing up your solutions! For instance: **you will only receive full credit if you provide full explanations**. Also, your **solutions should consist *solely* of complete sentences**. Simply providing the correct numerical solution does not suffice. See the *Mathematical Writing* appendix of the textbook for more tips.

Problem 1. At a Go tournament, there are eight players and four boards. In how many ways can the players sit down to play if

(a) we count who sits on which side of each board, but do not care about the ordering of the boards? (In this version, A vs B, C vs D, E vs F, G vs H is different from B vs A, C vs D, E vs F, G vs H but is the same as G vs H, E vs F, C vs D, A vs B.)

(b) We count the order in which pairs of players are seated at the boards, but do not care which side each player sits on? (Here A vs B, C vs D, E vs F, G vs H is the same as B vs A, C vs D, E vs F, G vs H but is different from G vs H, E vs F, C vs D, A vs B.)

Problem 2. In how many ways can King Arthur and his twelve knights (13 people, in all) sit down at the legendary Round Table in Camelot? (Since the table is round, we will not consider rotations of a given seating arrangement as different.)

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE WEDNESDAY WEEK 2

Problem 1. Let A and B be finite sets. Explain why

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Problem 2. Let A and B be sets with cardinalities $|A| = m$ and $|B| = n$. Suppose that $m \leq n$.

(a) What are the maximal and minimal values of $|A \cup B|$, and under what circumstances are these values achieved?

(b) What are the maximal and minimal values of $|A \cap B|$, and under what circumstances are these values achieved?

Problem 3. We have seen that there are 2^n subsets of a set A of cardinality n. We can use an *n-bit string* to encode such a subset. This is a length n word in the alphabet $\{0, 1\}$. Such an object looks like $b_{n-1}b_{n-1} \ldots b_0$ where each $b_i \in \{0, 1\}$, $0 \leq i \leq n - 1$. To turn a subset into a bit string, label the elements of A as $A = \{a_0, a_1, \ldots, a_{n-1}\}$; then for $B \in 2^A$, set

$$b_i = \begin{cases} 1 & \text{if } a_i \in B, \\ 0 & \text{if } a_i \notin B. \end{cases}$$

For instance, if $A = \{0, 1, 2, 3\}$ and $B = \{0, 2, 3\}$, then the associated bit string is 1101.

Given a bit string, we may treat it as a *binary representation* of a number. This associates the number

$$[b_{n-1}b_{n-2} \ldots b_1b_0]_2 = b_{n-1}2^{n-1} + b_{n-2}2^{n-2} + \cdots + b_1 2^1 + b_0 2^0$$

with the bit string $b_{n-1} \ldots b_1 b_0$. In the case of the bit string 1101, we have

$$[1101]_2 = 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 13.$$

(Of course, the final expression is a *decimal representation*: $13 = 1 \cdot 10^1 + 3 \cdot 10^0$.)

By turning a subset into a bit string and then a bit string into a number, we get a one-to-one correspondence between the subsets of A and the integers $0, 1, \ldots, 2^n - 1$. The following questions all refer to this numerical encoding of subsets of an arbitrary set A with $|A| = n$.

(a) What numbers correspond to subsets of cardinality one?

(b) What number corresponds to the subset $A \in 2^A$?

(c) What subsets correspond to even numbers?

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE FRIDAY WEEK 2

Problem 1. Your movie collection consists of five films directed by Werner Herzog, four films directed by Lana and Lilly Wachowski, and three films directed by Alejandro Jodorowsky. Give (good) examples of questions about your movie collection which have the following answers:

(a) $12 = 5 + 4 + 3$,
(b) $60 = 5 \cdot 4 \cdot 3$,
(c) $360 = 5 \cdot 4 \cdot 3 \cdot 3!$.

Problem 2. In the number theory chapter, we will study divisibility properties of integers. We say that an integer d *divides* an integer n when there exists an integer k such that $n = dk$. (As long as $n \neq 0$, this is the same as the fraction n/d being an integer.) In this problem, you may assume commons facts about integers, like the uniqueness of prime factorizations.

The number $169,400$ has prime factorization

$$169,400 = 2^3 \cdot 5^2 \cdot 7 \cdot 11^2.$$

Use the multiplicative counting principle to count the number of positive integers that divide $169,400$.

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE MONDAY WEEK 3

Problem 1. Let A be a nonempty finite set, let $E \subseteq 2^A$ be the collection of subsets of A of even cardinality, and let $O \subseteq 2^A$ be the collection of subsets of A of odd cardinality. Create an explicit bijective function $f\colon E \to O$ and conclude that $|E| = |O| = 2^{|A|-1}$. (You should define f by giving an explicit procedure one can perform to turn an element of E into an element of O. You should prove that f is bijective either by exhibiting a two-sided inverse, or by proving that f is injective and surjective.)

Problem 2. Let $f\colon A \to B$ be a function. Show that a function $g\colon B \to A$ such that $f \circ g = \mathrm{id}_B$ exists if and only if f is surjective. (Note that this is an "if and only if" proof. So there will be two parts to your proof: first suppose there is a function g with the stated properties, and show that it follows that f is surjective; next, suppose that f is surjective, and use that to prove that the appropriate function g exists.)

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE WEDNESDAY WEEK 3

Suppose we have an identity $E = F$ where E and F are two algebraic expressions that evaluate to the same integer (see the examples below). A *combinatorial* explanation for the identity $E = F$ requires identifying both E and F as solutions to counting problems and explaining why these counting problems should have the same solution. As an example, we give a proof of the identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

in the case where $n > k > 0$. (It is true for general n and k, but we will skip these trivial cases.)

Proof. Let $S = \{1, \ldots, n\}$. The left-hand side counts the k-subsets of S. Each k-subset of S is of exactly one of two types: (1) those that contain n, and (2) those that do not. To find the number of k-subsets of S, we can just count the numbers of each type and add. A subset of size k containing n, i.e, of type (1), is the same thing as a subset of $\{1, \ldots, n-1\}$ of size $k-1$ to which we then append n. Thus, there are $\binom{n-1}{k-1}$ subsets of type (1). A k-subset of S that does not contain n, i.e., of type (2), is the same as a subset of $\{1, \ldots, n-1\}$, and there are $\binom{n-1}{k}$ of these. $\square$

Problem 1. Consider the identity

$$\binom{n}{k} - \binom{n-3}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1}$$

for $n \geq 3$ and $n \geq k$.

(a) Suppose there is a set S of n people, and in that set, there are three special people a, b, and c. What is the left-hand side of the identity counting in the context of S and its three distinguished members?

(b) Provide a combinatorial proof of the identity by showing the thing you counted in part (a) can be counted a different way.

Problem 2. Give a combinatorial explanation of the following identity:

$$\binom{17}{5} = \binom{10}{0}\binom{7}{5} + \binom{10}{1}\binom{7}{4} + \binom{10}{2}\binom{7}{3} + \binom{10}{3}\binom{7}{2} + \binom{10}{4}\binom{7}{1} + \binom{10}{5}\binom{7}{0}.$$

Hint: you might think about coloring the elements of a set.

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE FRIDAY WEEK 3

Problem 1. Suppose that $f\colon A \to B$ is a surjective function. Define a relation $\asymp_f$ on A so that $a \asymp_f b$ if and only if $f(a) = f(b)$.

(a) Prove that $\asymp_f$ is an equivalence relation.
(b) Determine the number of equivalence classes under $\asymp_f$.

Problem 2. Suppose that we are playing a game in which we roll three six-sided dice (with sides labeled $1, 2, \ldots, 6$). Declare two rolls equivalent if their sums match. (Formally, a roll can be thought of as an ordered 3-tuple (a, b, c) where $a, b, c \in \{1, \ldots, 6\}$, and our relation is $(a, b, c) \sim (a', b', c')$ if and only if $a + b + c = a' + b' + c'$).)

(a) Prove that this is indeed an equivalence relation.
(b) Determine the number of equivalence classes.
(c) Are all of the equivalence classes of the same size?

Template for proving a relation is an equivalence relation.

Theorem. Define a relation $\sim$ on a set A by blah, blah, blah. Then $\sim$ is an equivalence relation.

Proof. *Reflexivity.* For each $a \in A$, we have $a \sim a$ since blah, blah, blah. Therefore, $\sim$ is reflexive.
Symmetry. Suppose that $a \sim b$. Then, blah, blah, blah. It follows that $b \sim a$. Therefore $\sim$ is symmetric.
Transitivity. Suppose that $a \sim b$ and $b \sim c$. Since blah, blah, blah, it follows that $a \sim c$. Therefore, $\sim$ is transitive.
Since $\sim$ is reflexive, symmetric, and transitive, it follows that $\sim$ is an equivalence relation. $\square$

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE FRIDAY WEEK 4

Problem 1. Use the binomial theorem to express 3^n as a sum of powers of two times binomial coefficients.

Problem 2. Let X be set of all subsets of size three from $\{1, \ldots, n+2\}$. For instance, if $n = 2$ we would have

$$X = \{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}\}.$$

In general, the number of such subsets is $|X| = \binom{n+2}{3}$. Each element of X consists of three numbers, which we list in order: $a < b < c$. For each integer b, let X_b be all subsets of $\{1, \ldots, n+2\}$ of the form $\{a, b, c\}$ for which $a < b < c$. We get a partition of X:

$$X = X_2 \amalg X_3 \amalg \cdots \amalg X_{n+1},$$

and hence

$$(\star) \qquad |X| = |X_2| + |X_3| + \cdots + |X_{n+1}|.$$

(a) Determine (with explanation, of course) the size $|X_b|$ for $b = 2, 3, \ldots, n+1$ in terms of b and n.
(b) Equation ($\star$) becomes what identity? (In other words, replace the quantities on the left and right in Equation ($\star$) with formulas. Note: to be sure of your answer, you should check it for small n on scratch paper.)

Note. Combinatorial identities often arise from partitioning a set. On your own, you may want to consider how the Problem 1 involves a partition.

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE MONDAY WEEK 5

Note. See the next page for a model proof by induction. Try to emulate it in your own work.

Problem 1. Use induction to prove that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for $n \geq 1$.

Problem 2. Suppose that n lines in the plane are drawn in such a fashion that no two are parallel and no three intersect in a common point. Prove that the plane is divided into precisely $\frac{n(n+1)}{2} + 1$ regions by the lines.

A typical induction proof

Proposition. For $n \geq 1$,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Proof. We will prove this by induction. First note that the statement holds when $n = 1$:

$$1 = \frac{1(1+1)}{2}.$$

Next, suppose the statement holds for some $n \geq 1$:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

It follows that

$$\begin{aligned} 1 + 2 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2}, \end{aligned}$$

and the result then holds for $n + 1$, too. Hence, the statement holds for all $n \geq 1$ by induction. $\square$

Note:

» The first sentence of the proof is obligatory. The reader needs to know you are about to give a proof by induction.

» It is good to explicitly state your induction hypothesis. In the above proof, it is the sentence starting "Next, suppose …" You are not claiming this statement is true! Your argument will be that *if* this statement is true, then something good happens (namely, the statement also holds for the case $n + 1$).

» Notice the easy-to-follow linear arrangement of equations following "It follows that". When you have a string of calculations, please try to use a similar form.

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE WEDNESDAY WEEK 5

Problem 1. In a room full of musicians, there are 10 who play guitar, 6 who play drums, and 6 who play bass. There are 3 who play both guitar and drums, 4 who play both guitar and bass, and 3 who play both drums and bass. There are 2 people who play all three instruments. How many musicians are there in total (with explanation)?

Problem 2. Use the principle of inclusion/exclusion to find how many numbers in $\underline{100} = \{1, 2, \ldots, 100\}$ are multiples of 2, 3, 5, or 7? (Show your work.)

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE FRIDAY WEEK 5

Problem 1. There are 10 people in a room of ages somewhere between 1- and 60-years old (inclusive).

(1) Use the pigeonhole principle to show there must be two distinct nonempty groups of people in the room such that the sum of each group's ages is the same.
(2) Prove there must be two *disjoint* nonempty groups of people in the room such that the sum of each group's ages is the same. [Hint: From the first part of this problem, you know there exist two distinct groups A and B whose age sums are equal. Start there.]

Problem 2. All of the integers 1 through 10 are placed in chairs around a circular table with 10 seats. Prove that there must be three neighbors whose sum is at least 17. [Hint: There are ten sets of neighbors as you go around the table. What number do you get if you add up these ten sets?]

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE MONDAY WEEK 6

Problem 1. How many graphs are there with vertex set $\{1, \ldots, 100\}$? Graphs are considered to be equal if they have the same edge sets. For instance, consider the case of graphs on the vertex set $\{1, \ldots, 4\}$. The following two graphs are different (e.g., the first has edge $\{1, 4\}$ and the second does not):

and the following are the same:

.

For this problem, we assume our graphs have no loops or multiple edges (i.e., each edge contains exactly two vertices). Also, note that the graph with no edges (consisting solely of isolated vertices) counts as a graph.

Problem 2. At every party, one can find two people who know the same number of other people at the party. (The property of "knowing" someone is assumed to be a symmetric relation but not reflexive.) Restate this assertion as a question about graphs, and prove it. [Hint: if there are n vertices in a graph, what is the list of possible vertex degrees? Use the pigeonhole principle.]

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE WEDNESDAY WEEK 6

Problem 1. A graph is *k-regular* if each of its vertices has degree k.

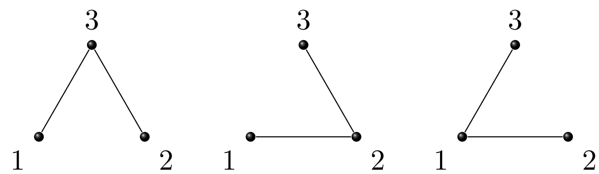
(a) Draw a 3-regular graph on 6 vertices.
(b) Prove that there are no 3-regular graphs on 5 vertices.

Problem 2. If G is a graph and e is an edge of G, define $G - e$ to be the graph obtained from G by removal of e (but not the endpoints of e). Recall that to say a graph is *connected* means that every pair of its vertices can be connected by a path.

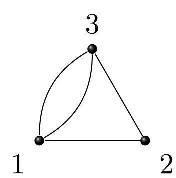
(a) Give an example of a connected graph G with an edge e such that $G - e$ is not connected.
(b) Suppose G is a connected graph and e is an edge of G that is part of a cycle. Prove that removal of e does not disconnect the graph. Your proof is required to start with the line: "Let u and v be vertices of G." It should then show there must be a path in $G - e$ connecting u and v.

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE FRIDAY WEEK 6

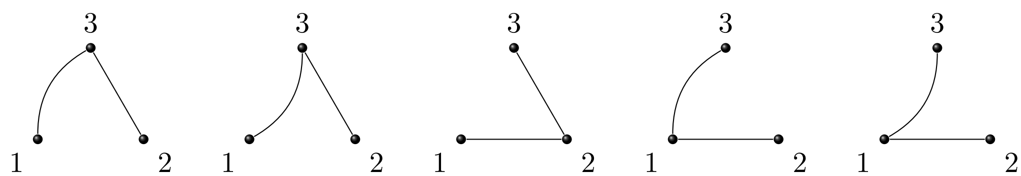
A *spanning tree* of a connected graph G is a subgraph T such that T is a tree and every vertex of G is on some edge of T. For instance, if G is the triangle with vertices $1, 2, 3$, then its spanning trees are:



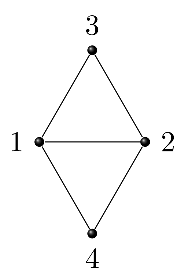
Recall that a *multigraph* is a graph in which multiple edges are allowed. For instance, the following graph has two edges connecting the vertices 1 and 3:



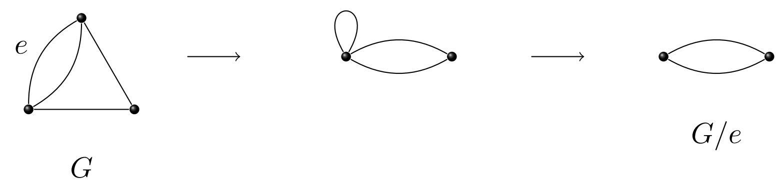
It has five spanning trees:



Problem 1. Draw all spanning trees of the following graph:



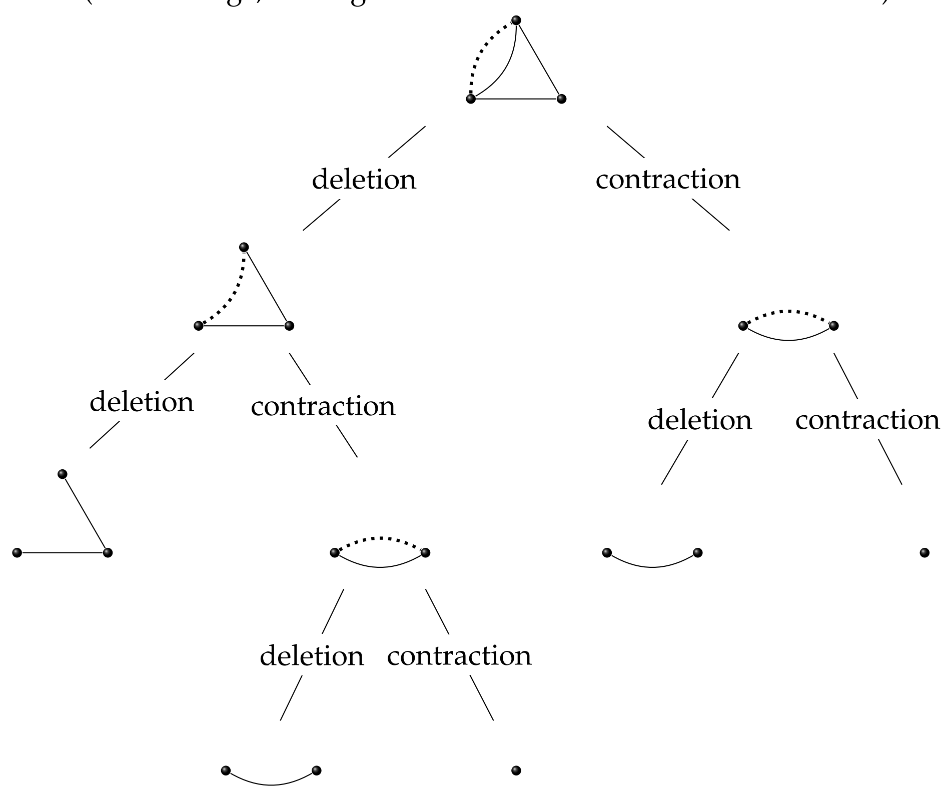
Problem 2 (deletion and contraction). Let G be a multigraph, and let e be an edge of G. Define $G-e$ to be the graph obtained from G by removing the edge e (but retaining the endpoints of e). Let G/e be the graph obtained from G by "contracting" the edge e. To contract e, remove e from G and then glue the endpoints of e together to make a single vertex from the two vertices. If there were multiple edges between the endpoints of e, loops will be formed, but for our purposes, we remove these loops as in the following:



(a) For an arbitrary connected multigraph G, choose an edge e such that $G - e$ is connected. Let $T(G)$, $T(G-e)$, and $T(G/e)$ denote the number of spanning trees of G, $G-e$, and G/e, respectively. The spanning trees of G come in two types: those that contain e and those that do not. Use that idea to prove

$$T(G) = T(G-e) + T(G/e).$$

(b) We can use the previous problem iteratively to count spanning trees. This is illustrated in the diagram below (at each stage, the edge chosen to delete and contract is dotted):



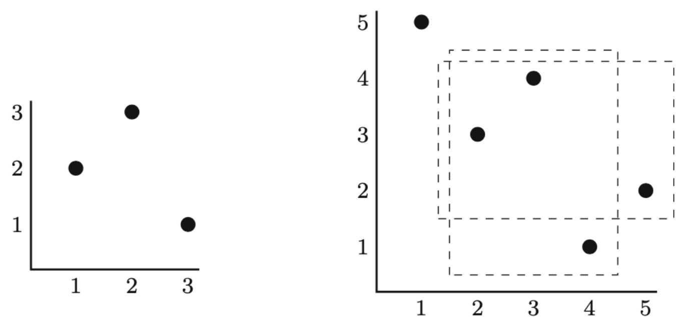
We stop in this deletion-contraction process when there are no edges left whose removal would leave a connected graph. Along the bottom, there are 5 trees (a single isolated vertex is considered to be a tree, too). The previous part of this problem implies there are 5 spanning trees of the original graph. These are the 5 spanning trees we saw earlier.

Make a similar diagram for the graph in Problem 1. (This diagram should verify the number of spanning trees you found earlier.)[1]

[1] The first part of Problem 2 implies the amazing fact that number of trees at the bottom of the diagram is independent of the choices of edges made in constructing the diagram!

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE MONDAY WEEK 7

A permutation $\pi \in \mathfrak{S}_n$ is called *231-avoiding* if there is no $1 \leq i < j < k \leq n$ such that $\pi(k) < \pi(i) < \pi(j)$. In other words, there are no $i < j < k$ such that $\pi(i), \pi(j), \pi(k)$ are in the same relative positions as $2, 3, 1$. It is easiest to check this condition visually with the graph (in the sense of functions) of the permutation. For instance, the permutation 53421 contains the pattern 231 in two ways, and thus is not 231-avoiding:

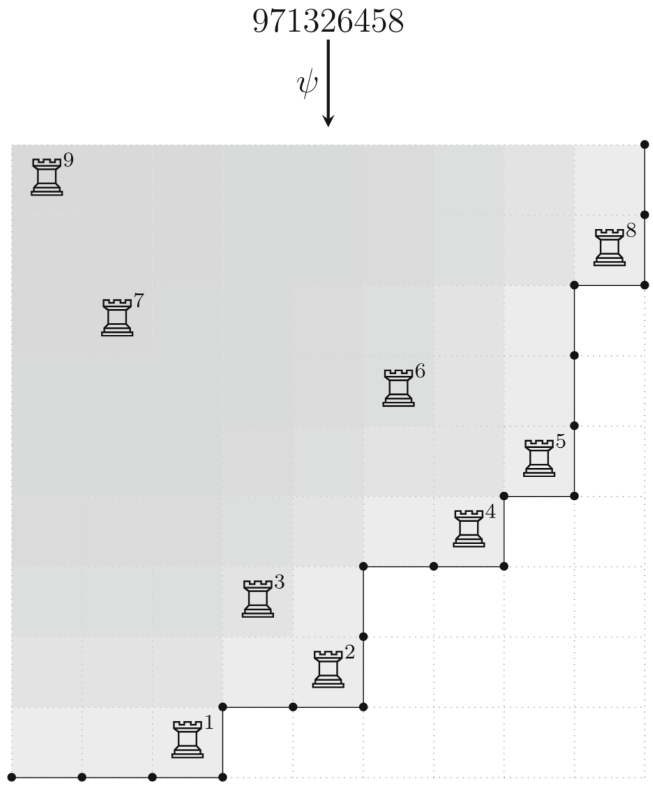


Let $\mathfrak{S}_n(231)$ denote the set of 231-avoiding permutations in $\mathfrak{S}_n$; let $s_n(231) := |\mathfrak{S}_n(231)|$.

Problem 1. List all of the 231-avoiding permutations for $n = 1, 2, 3, 4$ and compute the associated values of $s_n(231)$.

Problem 2. We now describe a bijection between 231-avoiding permutations and Dyck paths of length $2n$, providing a proof that $s_n(231) = C_n$.

(a) Given any permutation $\pi \in \mathfrak{S}_n$, think of the graph of π as a configuration of rooks on an $n \times n$ chess board. Shade all the squares that either contain a rook or are to the left of or above a rook. Let $\psi(\pi)$ denote the bottom-right boundary of the shaded region and prove that $\psi(\pi)$ is a Dyck path of length $2n$. *Hint*: If not, then there is some place where $\psi(\pi)$ goes above the line $y = x$. Then there is some $i \in [n]$ for which $\pi(i) > i$ and, since the path is non-decreasing in height, $\pi(j) > \pi(i)$ for all $j > i$. What is wrong with that? (On scratch paper, it might help to create examples of this situation to see what goes wrong.)

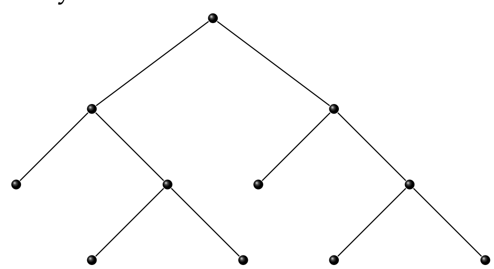


(b) It is possible that $\psi(\pi) = \psi(\sigma)$ even if $\pi \neq \sigma$. Give an example of such a σ in the case $\pi = 971326458$ (pictured above), and find $i < j < k$ such that $\sigma(k) < \sigma(i) < \sigma(j)$ (such i, j, k must exist, as discussed next).

(c) It turns out, though, that ψ gives a bijection between 231-avoiding permutations and Dyck paths: no two 231-avoiding permutations produce the same Dyck path, and every Dyck path arises by applying ψ to a 231-avoiding permutation. In this problem, let p be the Dyck path with corresponding balanced parenthesization (((()((())))))), and find the unique 231-avoiding permutation π such that $\psi(\pi) = p$.

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE FRIDAY WEEK 7

Problem 1. Consider the full binary tree T:

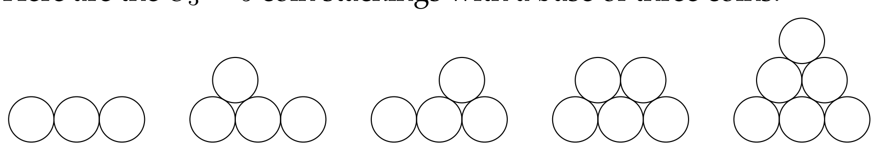


Using the bijections described in our text, find each of the following Catalan structures corresponding to T:

(a) a full parenthesization of the letters a, b, c, d, e, f ("full" means each multiplication is binary, i.e., involves two factors);
(b) a balanced parenthetical expression with five pairs of ()s;
(c) and a Dyck path of length ten.

Note: It is important to precisely use the conventions used in the handout. For example, reordering labels, etc., will not yield the same bijection.

Problem 2. Coin stackings form another set of Catalan structures (i.e., their count is given by Catalan numbers). Here are the $C_3 = 5$ coin stackings with a base of three coins:



(a) Draw the $C_4 = 14$ coin stackings with a base of 4 coins.
(b) Prove that the number of coin stackings with a base of n coins is C_n by describing a bijection between them and Dyck paths of length $2n$. [Hint: consider the region between a Dyck path and the diagonal.]

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE MONDAY WEEK 8

Your reading focused on *noncrossing partitions*. What about partitions in general, with no restriction on crossing? The following problems answer some combinatorial questions related to these structures.

Problem 1.

(a) Determine all of the partitions of the sets $[0], [1], [2], [3]$. (By definition, $[0] = \varnothing$. It has one partition—the empty partition.)

(b) Which partitions of $[4]$ cross (*i.e.*, which partitions are not noncrossing)?

Problem 2. Let B_n denote the number of partitions of $[n]$; this is called the n-th *Bell number*.

(a) Prove that for $n \geq 0$,

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

(*Hint*: Removing the block containing $n+1$ from a partition of $[n+1]$ leaves a partition of k elements for some $0 \leq k \leq n$.)

(b) Use this recurrence and Problem 1 to determine B_4 and B_5.

Problem 3. For $k, n \geq 0$, the *Stirling number of the second kind* $\left\{ {n \atop k} \right\}$ is defined to be the number of partitions of $[n]$ into k blocks. We have $\left\{ {0 \atop 0} \right\} = 1$ and $\left\{ {n \atop 0} \right\} = \left\{ {0 \atop n} \right\} = 0$ for $n > 0$.

(a) Prove that

$$\left\{ {n+1 \atop k} \right\} = k \left\{ {n \atop k} \right\} + \left\{ {n \atop k-1} \right\}$$

for $k, n > 0$. (Hint: partitions of $[n+1]$ into k blocks come in two types: those that contain $\{n+1\}$ as a singleton block, and those that do not.)

(b) Use this recursion to compute the Stirling numbers with $0 \leq k, n \leq 5$. You may want to arrange your results as you would arrange the binomial coefficients in Pascal's triangle. (*Note*: The relation $B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}$ will allow you to check your work.)

Remark. Famously, there is no "nice" formula for B_n. It is known (but you are not asked to prove) that

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n.$$

You can plug this into $B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}$ to get a rather un-nice closed formula for the Bell numbers.

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE WEDNESDAY WEEK 8

Problem 1. List all parking functions $p = (p_1, p_2, p_3, p_4)$ such that $p_i \leq 2$ for all i.

Problem 2. Which of the following are parking functions? Explain your reasoning.

(a) $(4, 1, 3, 3)$ (b) $(1, 1, 1)$ (c) $(4, 1, 3, 3, 4)$ (d) $(5, 2, 6, 2, 2, 5, 1)$.

Problem 3. For each of the following parking functions p, let q be the corresponding increasing parking function, found by sorting the elements of p, and then draw the Dyck path corresponding to q according to the method described in our text.

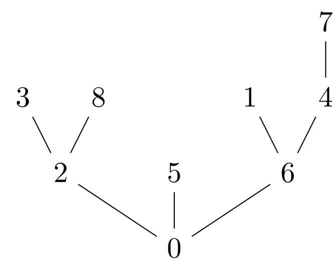
(a) $(1, 5, 3, 3, 1, 4)$ (b) $(1, 1, 1, 1)$ (c) $(4, 4, 1, 1, 2)$.

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE MONDAY WEEK 9

For the following problems, use the bijections developed in our text between parking functions, labeled Dyck paths, and labeled trees.

Problem 1. Find the labeled Dyck path and tree corresponding to the parking function $(3, 2, 6, 8, 5, 2, 1, 5)$.

Problem 2. Find the labeled Dyck path and the parking function corresponding to the tree



Problem 3. Let $p = (1, 1, \dots)$ be the parking function of length n with $p_i = 1$ for $i = 1, \dots, n$. Find the labeled Dyck path and tree corresponding to p.

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE FRIDAY WEEK 9

Problem 1. A subset X is chosen uniformly at random from the set $[n]$. (The word "uniform" here means that each subset is equally likely.)

(1) What is the probability that X has an even number of elements?
(2) Suppose $n \geq 2$. What is the probability that X contains 1 and n?
(3) Suppose $n \geq 10$. What is the probability that the smallest number in X is 10?

Problem 2. Suppose a bag contains balls numbered $1, 2, \ldots, 10$. Choose two balls from the bag.

(1) What is the probability the first ball is 5 and the second is 3 if the ball numbered 5 is not put back into the bag before drawing the second ball?
(2) What is the probability the first ball is 5 and the second is 3 if the ball numbered 5 is put back into the bag before drawing the second ball?

Problem 3.

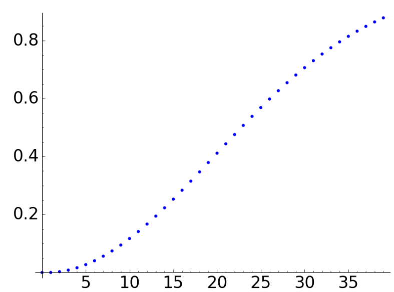
(1) What is the probability that a five-card poker hand contains exactly one ace?
(2) What is the probability that a five-card poker hand contains at least one ace?

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE MONDAY WEEK 10

Problem 1. Suppose a fair coin is flipped $n > 1$ times. We record the result at a string of length n in the letters H and T. For instance, if $n = 3$, then HTT says that the first flip was heads and the next two were tails. Our sample space S is, thus, the set of strings of length n consisting of the letters H and T, upon which we place the uniform distribution. Let A be the event that the first flip is heads, and let B be the event that the last flip is heads. Prove that these events are independent by computing the relevant probabilities and using the definition of independence.

Problem 2. This is the famous birthday problem. Suppose there are n people in a room. For simplicity, we will say that there are 365 possible birthdays (i.e., we will ignore leap years) and that each day is equally likely to be a birthday. To create the sample space, S, number the people from 1 to n, then the possible outcomes are the sequences of length n where the i-th element of the sequence is a possible birthday for person i. Let B be the event that at least two people in the room share a birthday.

(1) What is $|S|$?
(2) Consider the complementary event B^c, i.e., the event that no two people have the same birthday. Give an expression for B^c, and use it to find the probability $P(B^c)$. (The result will depend on n.)
(3) Use the previous result to give an expression for $P(B)$.
(4) A plot of $P(B)$ as a function of n looks like this:



Use a calculator of some sort to give decimal approximations, accurate to three decimal places, for $P(B)$ when $n = 22$ and $n = 23$. You should see that if the room contains at least 23 people, then the probability two people share the same birthday is more than one half.

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE WEDNESDAY WEEK 10

Problem 1. You have three coins. Two of the coins are fair: when flipped they are equally likely to land heads or tails. One coin, however, is weighted somehow so that its probability of landing heads is $3/4$.

(a) Choose one of the three coins uniformly at random and flip it. What is the probability the result is heads? For your solution, number the coins $1, 2, 3$ with coin 3 being the weighted one, and let A_i denote the event that coin i was chosen. Apply the generalized law of total probability (Theorem 177 our text).

(b) Choose one of the three coins at random and flip it. It lands heads. What is the probability that you chose the weighted coin? (Hint: Bayes' law.)

Problem 2. Reconsider the Monty Hall problem as stated in the group problems for Friday, Week 9, but where the game show has a bias for where it places the car so that $P(A) = 0.4$, $P(B) = 0.3$, and $P(C) = 0.3$. (In advance of your turn on the show, suppose that you studied taped shows to determined these propensities.) As in the group problems, let M_A, M_B, and M_C denote the events that "the host opens door A", "door B", and "door C", respectively.

(a) Show that no matter which door you pick, it makes sense to switch.
(b) Which door should you pick?
(c) What are your chances of eventually winning the car if you make that pick?

Note:

» The rules for Monty are the same: if you pick the door with the car, then Monty chooses between the remaining doors each with probability $1/2$, otherwise, Monty has only one choice: pick the door without the car.

» Up to symmetry, you just need to consider three cases: (i) you pick door A and Monty picks door B, (ii) you pick door B and Monty picks door A, and (iii) you pick door B and Monty picks door C. (If the all three of $P(A)$, $P(B)$, and $P(C)$ were different, there would be six cases to think about.)

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE FRIDAY WEEK 10

Problem 1. What is the expected value of the number of digits equal to 3 in a 4-digit positive integer? Write your solution as a fraction a/b in lowest terms. The sample space is

$$S = \{a_1a_2a_3a_4 : a_1 \in \{1, 2, \ldots, 9\}, a_2, a_3, a_4 \in \{0, 1, \ldots 9\}\}$$

[Hint: express the relevant random variable as a sum of simpler random variables, and use linearity of expectation.]

Problem 2. Let π be a permutation of $\underline{n}$. The index i is called an *exceedance* of π if $\pi(i) > i$. For instance, using the notation $\pi(1), \pi(2), \ldots, \pi(n)$ for a permutation π, the permutation $\pi = 3, 2, 4, 1$ has exceedance 2 since $\pi(1) = 3 > 1$ and $\pi(3) = 4 > 3$.

(a) Let X_i be the random variable on the set of permutations of $\underline{n}$ such that $X_i(\pi) = 1$ if i is an exceedance of π, and $X_i(\pi) = 0$, otherwise. What is the expected value, $E(X_i)$?

(b) How many exceedances does the average permutation of $\underline{n}$ have?

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE MONDAY WEEK 11

Problem 1. Ten fair six-sided dice are rolled. Five of the dice are red and five are green.

(a) Give a formula for the probability of rolling i sixes among the five red dice.
(b) What is the probability that there are the same number of sixes among the red dice as among the green dice? Use a calculator to estimate this probability with a decimal number accurate to at least 3 significant digits.

Problem 2. Five cards are dealt from a standard deck of 52 cards. What is the probability that two or more of the cards are aces? Use a calculator to estimate this probability with a decimal number accurate to at least 3 significant digits.

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE WEDNESDAY WEEK 11

Problem 1. Let r be the remainder when you divide b by a. Assume that $c \mid a$ and $c \mid b$. Prove that $c \mid r$.

Problem 2. Prove that for every integer a and positive integer n,

$$(a-1) \mid (a^n - 1).$$

(The proof can be quite short.)

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE FRIDAY WEEK 11

Problem 1. Use the Fundamental Theorem of Arithmetic to prove that if p is prime, a and b are integers, and $p|ab$, then either $p|a$ or $p|b$ (or both).

Problem 2. Let p be a prime and let a be an integer $1 \leq a \leq p-1$. Consider the numbers $a, 2a, 3a, \ldots, (p-1)a$. Use the division algorithm to write

$$ia = pq_i + r_i$$

with $0 \leq r_i < p$ and integers q_i for $1 \leq i \leq p-1$.

(a) Prove that $r_i > 0$ for each i.
(b) If $r_i = r_j$, show that $p|(i-j)a$, and explain why we can then conclude that $i = j$.
(c) Prove that $\{r_1, \ldots, r_{p-1}\} = \{1, 2, \ldots, p-1\}$.

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE WEDNESDAY WEEK 13

Problem 1. Use the Euclidean algorithm to compute the following (showing your work):

(a) $\gcd(20, 45)$ (b) $\gcd(247, 299)$ (c) $\gcd(51, 897)$.

Problem 2. Use the Euclidean algorithm to compute the $\gcd$ of 198 and 168 and then use back-substitution to find integers m and n such that

$$\gcd(198, 168) = 198m + 168n.$$

Show your work. Remember to use back-substitution and not the extended Euclidean algorithm.

Problem 3.

(a) Show that if n is positive integer of the form $4k + 3$ for some integer k, then n is not a perfect square. (Hint: Suppose $n = m^2$. We can then write $m = 4q + r$ for some $r \in \{0, 1, 2, 3\}$. Consider the remainders of the quantities $(4q)^2$, $(4q+1)^2$, $(4q+2)^2$, and $(4q+3)^2$ upon division by 4.)

(b) Show that no integer in the sequence

$$11, \quad 111, \quad 1111, \quad 11111, \quad \ldots$$

is a perfect square. [Hint: Use the fact that $111\ldots1111 = 111\ldots1108 + 3$.]

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE FRIDAY WEEK 13

Problem 1.

(a) Find the smallest positive integer n such that $7^n \equiv 1 \pmod{100}$.
(b) Use your solution to part (a) to find the last two digits of 7^{2020}. (You can use a computer to check your answer, but show how the solution can be derived easily by hand using part (a).
(c) (This part is optional and will not be graded.) What are the last two digits of

$$7^{7^{7^{\cdot^{\cdot^{\cdot^{7}}}}}}$$

in which the number of 7s appearing is 2020? Note $7^7 = 823543$ (or $43 \pmod{100}$), and $7^{7^7} = 7^{823543} \neq (7^7)^7 = 823543^7$.

Problem 2. Prove that if $a, b, c, m \in \mathbb{Z}$, $c \neq 0$, and $ac \equiv bc \pmod{mc}$, then $a \equiv b \pmod{m}$.

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE MONDAY WEEK 14

Problem 1. Find, with proof, the remainder of $9^{1260126012601260128}$ upon division by 14.

Problem 2.

(a) For $n = 10, 11$, and 12. List the fractions
$$\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}$$
after reducing each to lowest terms (canceling common factors in the numerator and denominator).

(b) Let n be a natural number and consider the quantity
$$\psi(n) = \sum_{d|n} \varphi(d)$$
which is the sum of the values $\varphi(d)$ where d ranges through all the positive divisors of n. What is $\psi(n)$? (Experiment, formulate a conjecture, and prove it.) Your solution should consist of a precise statement of your conjecture and a proof. The proof does not need to be elaborate. It can just be a statement of the general relevant phenomenon you observe in part (a).

Problem 3. There are integers n such that -1 has a square root in $\mathbb{Z}/n\mathbb{Z}$. To test this out, for each $n \in \{2, 3, \ldots, 13\}$, find all solutions $x \in \{0, 1, \ldots, n-1\}$ to the equation
$$x^2 \equiv -1 \pmod{n}.$$
You do not need to show your work. It may help to note that $-1 \equiv n - 1 \pmod{n}$.

Problem 4. Find all solutions $x \in \{0, 1, \ldots, n-1\}$ to the congruence $3x^2 - x + 1 \equiv 0 \pmod{n}$ for $n = 8$ and for $n = 9$. You do not need to show your work (but double-check your results!).

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE WEDNESDAY WEEK 14

As usual, show your work for the following problems.

Problem 1. Use Sunzi's Theorem to efficiently compute the congruence class of 17^2 modulo 35 as follows: First compute $17^2 \pmod 7$ and $17^2 \pmod 5$. Next, find the numbers in $\{0, 1, \ldots, 34\}$ that equal $17^2 \pmod 7$. Of these, which are equal to $17^2 \pmod 5$?

Problem 2. Describe *all* integer solutions to the system of congruences:

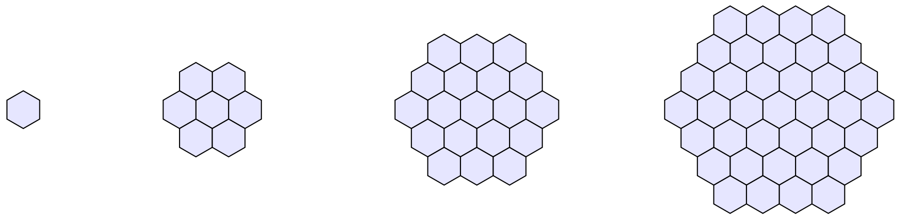
$$x = 1 \pmod 3$$
$$x = 2 \pmod 4$$
$$x = 3 \pmod 5.$$

Problem 3. Find integers $x, y \in \{0, 1, 2, \ldots, 7\}$ satisfying

$$x + 5y = 7 \pmod 8$$
$$3x + y = 1 \pmod 8.$$

MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE FRIDAY WEEK 14

Problem 1. Let h_n be the number of hexagons in a central packing of hexagons with $n+1$ layers. The first few values are pictured below:



$h_0 = 1$ $\quad$ $h_1 = 7$ $\quad$ $h_2 = 19$ $\quad$ $h_3 = 37$

Recall the first difference operator $\Delta[h]_n = h_{n+1} - h_n$.

(a) Notice that the pictures above nest in each other about the origin. Use that fact to draw pictures for the first differences $\Delta[h]$ and use the pictures to determine the sequence of second differences $\Delta^2[h]$.

(b) **Using the theory of difference operators from the text** and the results above, compute a polynomial $p(n)$ such that $p(n) = h_n$ for all $n \geq 0$.

Problem 2. Let F_n be the n-th Fibonacci number. Our text uses induction to show that

$$F_{2n+1} = F_n^2 + F_{n+1}^2. \tag{1}$$

You are now asked to give a combinatorial proof using tilings of checkerboards. Let a_n be the number of ways of tiling a $2 \times n$ checkerboard with 2×1 dominoes. At the end of the section on induction, our text shows that $a_n = F_{n+1}$.

Rewrite equation (1) in terms of appropriate a_i and prove the resulting (equivalent) formula by counting tilings of a $2 \times 2n$ checkerboard. (Hint: Our checkboard has two halves, each of size $2 \times n$. Consider how dominoes in a tiling behave at the middle where these two halves meet. There are only two possibilities: there is a pair of dominoes straddling the two halves, or no domino straddles the center. Given that each half of our chessboard contains an even number of dominoes, it is impossible for only one domino to straddle the center.)