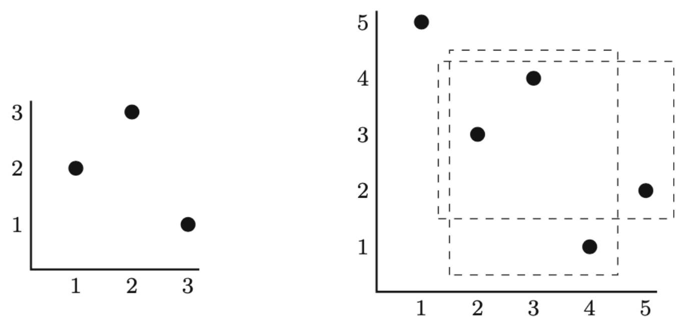
MATH 113: DISCRETE STRUCTURES
HOMEWORK DUE MONDAY WEEK 7

A permutation $\pi \in \mathfrak{S}_n$ is called *231-avoiding* if there is no $1 \leq i < j < k \leq n$ such that $\pi(k) < \pi(i) < \pi(j)$. In other words, there are no $i < j < k$ such that $\pi(i), \pi(j), \pi(k)$ are in the same relative positions as $2, 3, 1$. It is easiest to check this condition visually with the graph (in the sense of functions) of the permutation. For instance, the permutation 53421 contains the pattern 231 in two ways, and thus is not 231-avoiding:

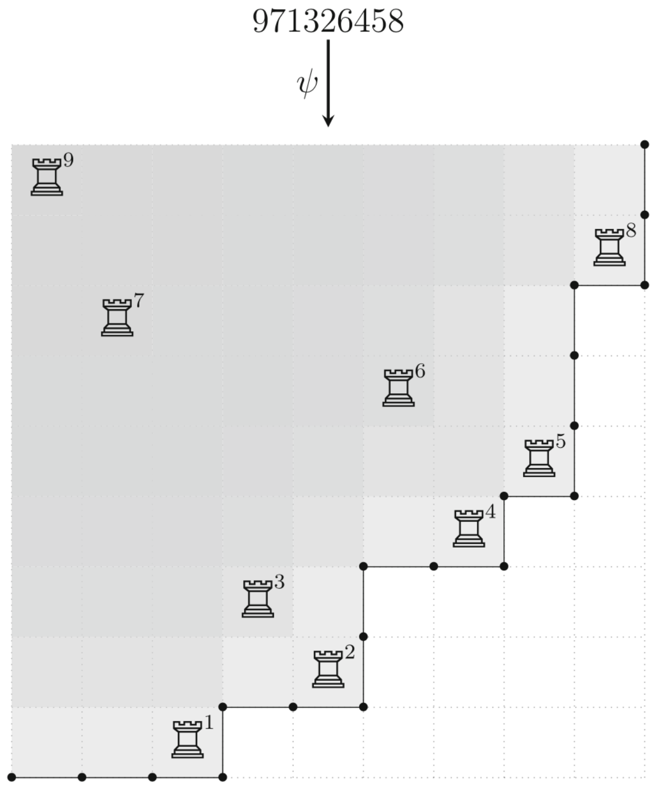


Let $\mathfrak{S}_n(231)$ denote the set of 231-avoiding permutations in $\mathfrak{S}_n$; let $s_n(231) := |\mathfrak{S}_n(231)|$.

Problem 1. List all of the 231-avoiding permutations for $n = 1, 2, 3, 4$ and compute the associated values of $s_n(231)$.

Problem 2. We now describe a bijection between 231-avoiding permutations and Dyck paths of length $2n$, providing a proof that $s_n(231) = C_n$.

(a) Given any permutation $\pi \in \mathfrak{S}_n$, think of the graph of π as a configuration of rooks on an $n \times n$ chess board. Shade all the squares that either contain a rook or are to the left of or above a rook. Let $\psi(\pi)$ denote the bottom-right boundary of the shaded region and prove that $\psi(\pi)$ is a Dyck path of length $2n$. *Hint*: If not, then there is some place where $\psi(\pi)$ goes above the line $y = x$. Then there is some $i \in [n]$ for which $\pi(i) > i$ and, since the path is non-decreasing in height, $\pi(j) > \pi(i)$ for all $j > i$. What is wrong with that? (On scratch paper, it might help to create examples of this situation to see what goes wrong.)



(b) It is possible that $\psi(\pi) = \psi(\sigma)$ even if $\pi \neq \sigma$. Give an example of such a σ in the case $\pi = 971326458$ (pictured above), and find $i < j < k$ such that $\sigma(k) < \sigma(i) < \sigma(j)$ (such i, j, k must exist, as discussed next).

(c) It turns out, though, that ψ gives a bijection between 231-avoiding permutations and Dyck paths: no two 231-avoiding permutations produce the same Dyck path, and every Dyck path arises by applying ψ to a 231-avoiding permutation. In this problem, let p be the Dyck path with corresponding balanced parenthesization $(((()((())))))$, and find the unique 231-avoiding permutation π such that $\psi(\pi) = p$.