PROBLEM 1. Consider the sequence $a_0, a_1, \ldots$ defined by the recurrence

$$a_0 = 0, \quad a_1 = 1, \quad \text{and} \quad a_n = 5a_{n-1} - 6a_{n-2} \text{ for } n \geq 2.$$

(i) Write out the terms of (a_n) until you get to 2059.

(ii) Check that for $a \neq b$,

$$\frac{x}{(1-ax)(1-bx)} = \frac{1}{a-b}\left(\frac{1}{1-ax} - \frac{1}{1-bx}\right).$$

(iii) In the text, we used generating functions to find a closed form for the Fibonacci numbers. Apply a similar procedure to $f(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots$, the generating function for (a_n), to find a closed form for (a_n).

PROBLEM 2. Let $f(x) = \sum_{i=0}^{\infty} b_i x^i$ be the generating function for the sequence $b_0, b_1, \ldots$.

(i) Let $g(x) = (1-x)f(x)$. Then $(g(x) - b_0)/x$ is the generating function for which sequence?

(ii) Let $h(x) = \frac{f(x)}{1-x}$. Then $h(x)$ is the generating function for which sequence?

(iii) Apply the previous result to $h(x) = 1/(1-x)$ to find the sequence whose generating function is $1/(1-x)^2$.

(iv) Find the sequence whose generating function has closed form $\frac{1+x+x^2}{(1-x)^2}$ by multiplying $1 + x + x^2$ by the series for $1/(1-x)^2$.