PROBLEM 1. Suppose that $a \in \mathbb{R}^{\mathbb{N}}$ is a polynomial sequence of degree 4. Use the following table of differences to determine a formula for a_n.

a_n	0		0		4		12		72		$\cdots$		
$\Delta[a]_n$		0		4		8		60		$\cdots$			
$\Delta^2[a]_n$			4		4		52		$\cdots$				
$\Delta^3[a]_n$				0		48		$\cdots$					
$\Delta^4[a]_n$					48		$\cdots$						

PROBLEM 2. With your group, choose a "random" polynomial p of degree at most 5. Prepare a table of the values $p(n)$ for $n = 0, 1, \ldots, 6$. Swap tables of values with another group and then reconstruct each others polynomials. Here is an example of https://sagecell.sagemath.org/ code that might help:

```
f(x) = x^5 + 5*x^4 - 2*x^3 + x^2 + 3*x + 7
[f(i) for i in [0..6]]
```

Output:

```
[7, 15, 113, 619, 2211, 6047, 13885]
```

PROBLEM 3.

(i) Fix $r \geq 0$. For $n \geq 0$ define the sequence whose n-th term is $a_n = \sum_{k=0}^{n} k^r$. Prove that $(a_n)_{n=0}^{\infty}$ is a degree $r+1$ polynomial sequence. (Hint: our reading implies that is suffices to show that $\Delta[a]_n$ is a polynomial sequence of degree r.)

(ii) Consider the case $r = 3$. Use a table of differences to determine a polynomial expression for

$$a_n = \sum_{k=0}^{n} k^3.$$

Fun fact: you can write that polynomial as the square of a single binomial coefficient involving n.

PROBLEM 4.

(i) Prove that $3 \mid n^3 + 2n$ for all $n \in \mathbb{N}$ in two ways: (1) show $n^3 + 2n = 0 \pmod 3$ by checking all possibilities for $n \pmod 3$, and (2) by using a table of differences to write $n^3 + 2n$ as a sum of $\binom{n}{k}$s.

(ii) Suppose that f is a numerical polynomial of degree d. Prove that d divides $f(n)$ for all $n \in \mathbb{N}$ if and only if d divides $\Delta^k[f]_0$ for all $k \geq 0$. (Hints: one direction follows immediately from Theorem 89, and the other follows by considering what the table of differences would look like if d divides $f(n)$ for all n.)

(iii) Use the part (ii) to determine the largest number dividing $n^5 - n$ for all $n \in \mathbb{N}$?