PROBLEM 1. Show that if A and B are independent, then so are their complements A^c and B^c. Steps:

(i) State, mathematically, what it is you need to show and what you get to assume.
(ii) Use the standard identity for sets $A^c \cap B^c = (A \cup B)^c$, and the facts, easily derivable from the three axioms for a probability distribution, that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ and that $P(C^c) = 1 - P(C)$ for any event C.

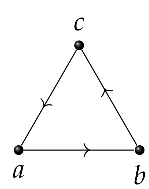
PROBLEM 2. Roll two fair 6-sided dice, one red and one blue. Let A be the event that the red die is odd, let B be the event that the blue die is odd, and let C be the event that the sum of the dice is odd.

(i) Show that each of the three pairs of these events is independent.
(ii) Show that $P(A \cap B \cap C) \neq P(A)P(B)P(C)$.
(iii) Why is this example interesting?

Challenge

Challenge problems are optional and should only be attempted after completing the previous problems.

There are n players in a Go tournament in which each pair of participants play each other. If $n > 2$, then it is possible that each person has lost to someone. For instance, suppose the three players are a, b, c, it could be that a beat b, and b beat c, and c beat a. We can picture this situation using the following directed graph in which a directed edge points from the winning to the loser:



If n is large enough, is it possible that for every *pair* $\{x, y\}$ of players, there a person who has beat both x and y? If so, what is the smallest n for which that is possible? Instead of answering that question (which you can think about later), in this problem we will go one further and use probability theory to show that if n is large enough, it is possible that at the end of the tournament, for every collection of *three* players there exists another player who has beaten them all. Note the curious fact that our proof does not explicitly describe any specific instance of this occurrence.

(i) Suppose that the outcome of each game is random. (Perhaps the players are lazy and flip a coin to decide the winner.) Fix a 3-subset $\{x, y, z\}$ of players and some player w not in $\{x, y, z\}$.

What is the probability that w wins against x, y, and z? What is the probability that w loses against at least one of x, y, z?

(ii) Suppose we have another player w' different from w, x, y, and z. Are the results of w''s matches against x, y, z independent of the results of w's matches?

(iii) How many players can appear in the role of w? What is the probability that each of them loses against at least one of x, y, z?

(iv) Explain why, in general, for any probability space with events A and B and probability distribution P, we have $P(A \cup B) \leq P(A) + P(B)$? Exactly when does equality hold?

(v) Use your answers to (iii) and (iv) and the fact that there are $\binom{n}{3}$ 3-subsets of $[n]$ to produce an upper bound on the probability that for at least one 3-subset $\{x, y, z\}$, no player beats x, y, and z simultaneously, (equivalently, for at least one 3-subset, everyone not in the subset loses to at least one of x, y, or z, as in part (iv))?

(vi) What does it mean if your upper bound from (v) is less than 1? Use a computer to determine if there are n for which this happens.