PROBLEM 1. Let NC_n denote the number of noncrossing partitions of $[n]$ for $n \geq 0$ (taking $NC_0 := 0$). In the text, you saw a direct bijection exhibiting that $NC_n = C_n$. Here, we reprove this via the Catalan recurrence:

$$C_0 = 1 \quad \text{and} \quad C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \text{ for } n \geq 0.$$

Claim: $NC_n = C_n$ for all $n \geq 0$.

Proof. We will prove this by induction.

(i) The base case holds since ...

(ii) Next fix some $n \geq 0$ and suppose the result holds for all k such that $0 \leq k \leq n$. Consider any noncrossing partition P of $[n+1]$. (At this point, you should draw a nontrivial example in order to follow along.) There is some block B of P that contains $n+1$. Let ℓ be the next smallest number in B. Why is it the case that every other block of the partition consists either of numbers that are smaller than ℓ or of numbers that are bigger than ℓ? (Refer to your example to think about this question.)

(iii) For $1 \leq \ell \leq n$, let X_ℓ be the collection of all noncrossing partitions of $[n+1]$ such that the block containing $n+1$ has second smallest number ℓ. Also, let X_0 be the set of partitions of $[n+1]$ containing the block $\{n+1\}$. Using induction and the MCP, argue that $|X_\ell| = C_\ell C_{n-\ell}$.

(iv) Use the Catalan recurrence to finish the proof.

□

PROBLEM 2. The end of the video lecture formulates a direct bijection between Dyck paths of length $2n$ and noncrossing partitions of $[n]$.

(i) Letting $n = 8$, create a Dyck path of length $2n$ and use the method of the video lecture to find the corresponding noncrossing partition.

(ii) Create a noncrossing partition of [7], and find its corresponding Dyck path using the method of the video lecture.

(iii) Call a transition from an east step to a north step in a Dyck path a *valley*. Verify in your examples that the number of valleys in a Dyck path corresponds to the number of blocks in the associated partition.

Challenge

Challenge problems are optional and should only be attempted after completing the previous problems.

Show that the number of noncrossing partitions of $[n]$ with exactly k blocks is the *Narayana number*

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

This is also the number of Dyck paths of length $2n$ with exactly k valleys. Conclude that

$$C_n = \sum_{k=1}^{n} N(n,k).$$