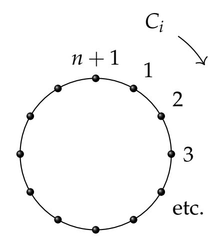
PROBLEM 1.

(i) In turn, each person in your group should make up a parking function p of length five. The rest of the group should then check that p is a parking function by (i) using the definition of a parking function (i.e., the list of preferences allows every car to park), and (ii) sorting p to get an increasing parking function and comparing with $(1, 2, 3, 4, 5)$.

(ii) Do the same, but now each person should create a non-parking function $p = (p_1, \cdots, p_5)$ such that $1 \leq p_i \leq 5$. Again, check each p in two ways.

PROBLEM 2 (Circular parking functions). Consider a variation of the protocol for parking cars discussed in class. There are still n cars, $C_1, \ldots, C_n$, but this time there is one extra parking space, numbered $n+1$, and the spaces are arranged in a circle. Car C_i prefers to park in space $p_i \in \{1, \ldots, n+1\}$. Other than that, the rules are essentially the same: starting just before space 1, each car in turn drives around the circle to its preferred spot and parks there if possible. Otherwise, it drives on to the next available spot. Since the spaces are arranged in a circle and there are more spaces than cars, each car will eventually park. The preference list $p = (p_1, \ldots, p_n)$ is called a *circular parking function*.



(a) Find the resulting positions of the cars $C_1, \cdots, C_5$ parking according to the following circular parking functions:

$$\begin{array}{llllll} \text{(i)} & (3,2,1,3,5) & \text{(ii)} & (4,2,4,2,1) & \text{(iii)} & (4,1,1,3,5) \\ \text{(iv)} & (5,6,1,3,3) & \text{(v)} & (2,2,5,4,5) & \text{(vi)} & (6,6,6,6,6). \end{array}$$

Recall that there are now six parking spaces.

(b) Why are there $(n+1)^n$ circular parking functions?

(c) Each circular parking function leaves one space empty. For $i = 1, \cdots, n+1$, let P_i be the set of circular parking functions that leave space i empty. If P is the set of all circular parking functions, then we have a partition:

$$P = P_1 \amalg \cdots \amalg P_{n+1}.$$

It turns out that each P_i has the same cardinality. Given that, what is $|P_i|$ for each i?

(d) Where have you seen the elements in P_{n+1} before?

(e) Based on the above results, argue that the number of ordinary parking functions of length n is the number of labeled trees on $n+1$ vertices.

Challenge

Challenge problems are optional and should only be attempted after completing the previous problems.

Using the notation from the previous problem, prove that each P_i has the same cardinality.