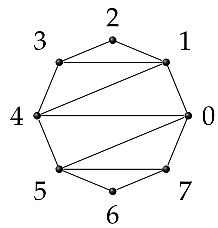
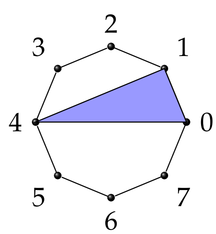
PROBLEM 1. A *triangulation* of a convex $n$-gon is a collection of non-intersecting diagonals (line segments between non-adjacent vertices) that break the $n$-gon into triangles. For instance, the following is a triangulation of an 8-gon:




(i) Let $T_n$ denote the number of triangulations of an $n$-gon. Draw all triangulations of convex $n$-gons for $n = 3, 4, 5, 6$, and make a conjecture for the value of $T_n$, in general.

(ii) Prove your conjecture. *Hint*: Label the vertices of the $n$-gon by $0, 1, \cdots, n-1$, cyclically. Given any triangulation, exactly one triangle will contain the edge 01. Use this triangle as the basis for a recursion. For example, for each $k = 2, 3, \ldots, 7$, how many triangulations are there of the 8-gon which contain the triangle $01k$? For $k = 4$, we would be asking how many ways are there of completing the triangulation of the figure below:



PROBLEM 2. From our reading, we know that full binary trees with $n+1$ leaves and balanced parenthesizations of length $2n$ are counted by the Catalan number $C_n$. The reading also includes a description of a direct bijection between these two structures. Briefly, given a full binary tree, label the left edges with '(' on their left and ')' on their right. Start at the root of the tree and start walking down the leftwards edge; keep the tree on your left and record the labels as you pass them. The resulting is the balanced parenthesization corresponding to the binary tree.

Prove that the process described above works, i.e., that it provides a bijection. It is recommended that you follow these steps:

(i) Draw several full binary trees and produce the resulting balanced parenthesizations.
(ii) Why is the resulting parenthesization always balanced?
(iii) Now create several balanced parenthesizations, and find their corresponding binary trees.
(iv) Describe an algorithm (or function) for turning a balanced parenthesization into a full binary tree which is inverse to our process of turning binary trees into balanced parenthesizations.

## *Challenge*

Challenge problems are optional and should only be attempted after completing the previous problems.

Produce a direct bijection between triangulations of a convex $n$-gon and full binary trees with $n-1$ leaves. Show that diagonal flips of edges in a triangulation correspond to tree "rotations". (A *diagonal flip* transforms a pair of edge-sharing triangles from ⧄ into ⧅.) [Hint: Identify the vertices of the tree with the edges of the triangulation, including the edges of the $n$-gon. Fix one of the edges of the $n$-gon to be the root of the tree. Which two vertices (edges of the triangulation) should be adjacent to the root? How can you continue to grow the tree from there?] Carry out the procedure for the triangulation of the octagon given at the start of this problem set. What happens to the binary tree if you flip the triangulation of the quadrilateral 0134 in that example?