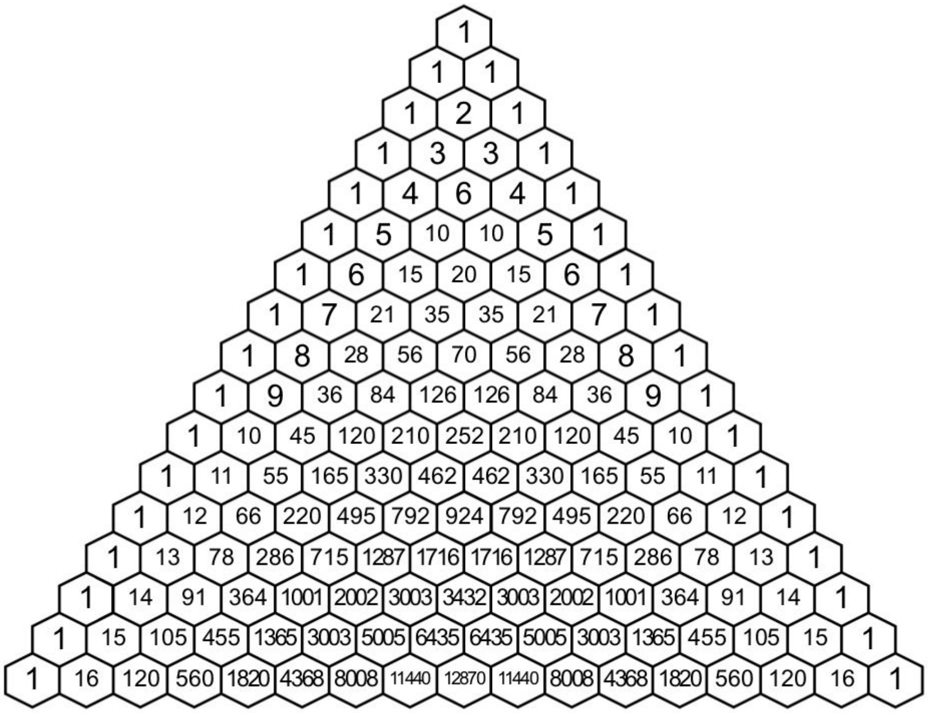
For reference, here is a copy of Pascal's triangle:



and here are two versions of the binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

PROBLEM 1.

(i) Compute the sums

$$\binom{0}{0}^2$$

$$\binom{1}{0}^2 + \binom{1}{1}^2$$

$$\binom{2}{0}^2 + \binom{2}{1}^2 + \binom{2}{2}^2$$

$$\binom{3}{0}^2 + \binom{3}{1}^2 + \binom{3}{2}^2 + \binom{3}{3}^2$$

$$\binom{4}{0}^2 + \binom{4}{1}^2 + \binom{4}{2}^2 + \binom{4}{3}^2 + \binom{4}{4}^2$$

$$\binom{5}{0}^2 + \binom{5}{1}^2 + \binom{5}{2}^2 + \binom{5}{3}^2 + \binom{5}{4}^2 + \binom{5}{5}^2$$

by hand and develop a conjecture regarding the value of

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n-1}^2 + \binom{n}{n}^2.$$

(ii) Use the binomial theorem to prove your conjecture. [*Hint*: We have the identity $(1+x)^{2n} = (1+x)^n(1+x)^n$. Therefore, if we expand either side and find the coefficient of x^n, we will get the same number. Use the binomial theorem to find the coefficient of x^n in $(1+x)^{2n}$. Next apply the binomial theorem to $(1+x)^n$ and use the result to find the coefficient of x^n in $(1+x)^n(1+x)^n$.]

(iii) Give a combinatorial argument proving your conjecture. [*Hint*: Split a set of size $2n$ into two pieces of size n, and then start building size n subsets of the original set.]

PROBLEM 2. The book claims that

$$\sum_{\ell=k}^{n} \binom{\ell}{k} = \binom{n+1}{k+1}$$

for all $k, n \in \mathbb{Z}$.

(i) Write out the above identity for the case $n = 5$ and $k = 2$.

(ii) Highlight the terms involved in this identity for various k and n on Pascal's triangle; explain why it is known as the *hockey stick identity*. (Recall that the row's of Pascal's triangle are indexed starting with $n = 0$.)

(iii) Let X be the set of subsets of $[n+1]$ of cardinality $k+1$, and let

$$X_a := \{A \in X \mid a \text{ is the first element of } [n+1] \text{ in } A\}$$

for $a = 1, 2, \dots, n-k$. Check that the X_i partition X:

$$X = X_1 \amalg X_2 \amalg \cdots \amalg X_{n-k+1}.$$

(Is each $(k+1)$-subset of $[n+1]$ in exactly one X_i? We do we stop with the index $n-k+1$?)

(iv) Determine the cardinality of X_a in terms of n, k, and a. Use this and (ii) to give a combinatorial proof of the hockey stick identity.

Challenge

Challenge problems are optional and should only be attempted after completing the previous problems.

How many ways are there to write a nonnegative integer m as a sum of r positive integer summands? (We decree that the order of the addends matters, so $3+1$ and $1+3$ are two different representations of 4 as a sum of 2 nonnegative integers.) Develop a conjecture and prove it.

Challenge

Challenge problems are optional and should only be attempted after completing the previous problems.

Answer the variation of the previous challenge problem in which we allow *nonnegative* integer summands.