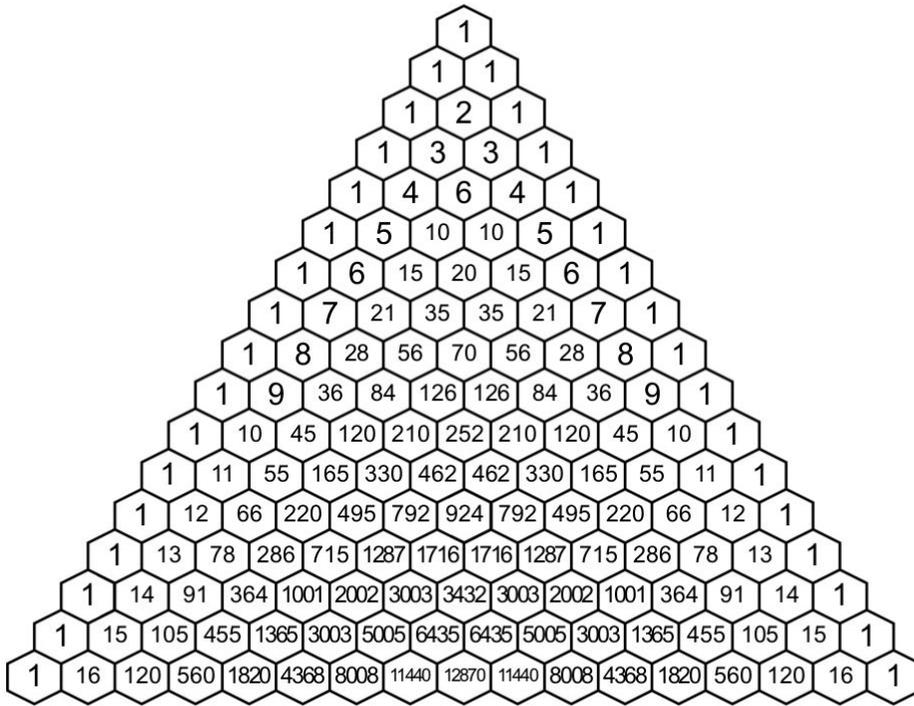


For reference, here is a copy of Pascal's triangle:



and here are two versions of the binomial theorem:

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

$$(1 + x)^n = \sum_{k=0}^n \binom{n}{k} x^k.$$

PROBLEM 1.

(i) Compute the sums

$$\begin{array}{c}
 \binom{0}{0}^2 \\
 \binom{1}{0}^2 + \binom{1}{1}^2 \\
 \binom{2}{0}^2 + \binom{2}{1}^2 + \binom{2}{2}^2 \\
 \binom{3}{0}^2 + \binom{3}{1}^2 + \binom{3}{2}^2 + \binom{3}{3}^2 \\
 \binom{4}{0}^2 + \binom{4}{1}^2 + \binom{4}{2}^2 + \binom{4}{3}^2 + \binom{4}{4}^2 \\
 \binom{5}{0}^2 + \binom{5}{1}^2 + \binom{5}{2}^2 + \binom{5}{3}^2 + \binom{5}{4}^2 + \binom{5}{5}^2
 \end{array}$$

by hand and develop a conjecture regarding the value of

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n-1}^2 + \binom{n}{n}^2.$$

SOLUTION: we compute

$$\begin{array}{c}
 \binom{0}{0}^2 = 1 \\
 \binom{1}{0}^2 + \binom{1}{1}^2 = 2 \\
 \binom{2}{0}^2 + \binom{2}{1}^2 + \binom{2}{2}^2 = 6 \\
 \binom{3}{0}^2 + \binom{3}{1}^2 + \binom{3}{2}^2 + \binom{3}{3}^2 = 20 \\
 \binom{4}{0}^2 + \binom{4}{1}^2 + \binom{4}{2}^2 + \binom{4}{3}^2 + \binom{4}{4}^2 = 70 \\
 \binom{5}{0}^2 + \binom{5}{1}^2 + \binom{5}{2}^2 + \binom{5}{3}^2 + \binom{5}{4}^2 + \binom{5}{5}^2 = 252
 \end{array}$$

Suspiciously and amazingly, these appear in the center column of Pascal's triangle as the numbers of the form $\binom{2n}{n}$. We conjecture that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n-1}^2 + \binom{n}{n}^2 = \binom{2n}{n}$$

and note that this matches the cases computed above.

- (ii) Use the binomial theorem to prove your conjecture. [Hint: We have the identity $(1+x)^{2n} = (1+x)^n(1+x)^n$. Therefore, if we expand either side and find the coefficient of x^n , we will get the same number. Use the binomial theorem to find the coefficient of x^n in $(1+x)^{2n}$. Next apply the binomial theorem to $(1+x)^n$ and use the result to find the coefficient of x^n in $(1+x)^n(1+x)^n$.]

SOLUTION: Let x be a variable. By the binomial theorem

$$(1+x)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} x^i.$$

In particular, the coefficient of x^n in this polynomial is $\binom{2n}{n}$.

We also have $(1+x)^{2n} = (1+x)^n(x+1)^n$, and applying the binomial theorem to each factor results in

$$(1+x)^{2n} = \left(\sum_{i=0}^n \binom{n}{i} x^i \right) \left(\sum_{j=0}^n \binom{n}{j} x^{n-j} \right).$$

When we expand this product, we get a term contributing to x^n when $i+n-j = n$, i.e. when $i = j$. Thus the coefficient of x^n is $\sum_{i=0}^n \binom{n}{i}^2$, and this must equal our alternate computation of the coefficient, $\binom{2n}{n}$.

- (iii) Give a combinatorial argument proving your conjecture. [Hint: Split a set of size $2n$ into two pieces of size n , and then start building size n subsets of the original set.]

SOLUTION: Suppose $|A| = 2n$ and then color half its elements blue and half its elements red. (We can do that!) To get a size n subset of A , we can choose a blue elements and b red elements where $a+b = n$. For fixed a , there are $\binom{n}{a}\binom{n}{b}$ ways to do this. Since $b = n - a$, we have $\binom{n}{b} = \binom{n}{n-a} = \binom{n}{a}$, and so $\binom{n}{a}\binom{n}{b} = \binom{n}{a}^2$. Letting a vary from 0 to n , we see that in sum we have

$$\begin{aligned} \binom{2n}{n} &= \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{n}\binom{n}{0} \\ &= \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2, \end{aligned}$$

as desired.

PROBLEM 2. The book claims that

$$\sum_{\ell=k}^n \binom{\ell}{k} = \binom{n+1}{k+1}$$

for all $k, n \in \mathbb{Z}$.

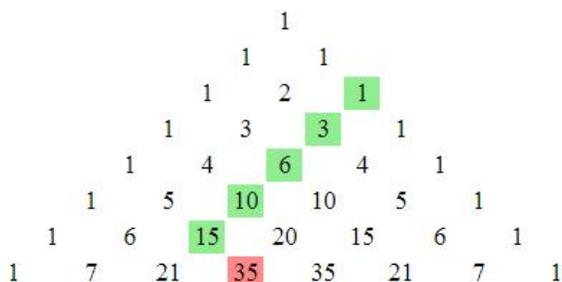
- (i) Write out the above identity for the case $n = 5$ and $k = 2$.

SOLUTION: We have

$$\sum_{\ell=2}^5 \binom{\ell}{2} = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} = \binom{6}{3}.$$

- (ii) Highlight the terms involved in this identity for various k and n on Pascal's triangle; explain why it is known as the *hockey stick identity*.

SOLUTION: The hockey stick identity says that the terms in the "stick" add up to the "blade" term in the following picture:



- (iii) Let X be the set of subsets of $[n + 1]$ of cardinality $k + 1$, and let

$$X_a := \{A \in X \mid a \text{ is the first element of } [n + 1] \text{ in } A\}$$

for $a = 1, 2, \dots, n - k$. Check that

$$X = X_1 \amalg X_2 \amalg \dots \amalg X_{n-k+1}.$$

(Is each $(k + 1)$ -subset of $[n + 1]$ in exactly one X_i ? We do we stop with the index $n - k + 1$?)

SOLUTION: Clearly each X_a is a subset of X and they are disjoint since elements of X_a and X_b have different first elements when $a \neq b$. Additionally, every element of X has a first element between 1 and $n - k + 1$ (no higher since the cardinality is $k + 1$). This shows that $\{X_1, \dots, X_{n-k+1}\}$ is a partition of X .

- (iv) Determine the cardinality of X_a in terms of n , k , and a . Use this and (ii) to give a combinatorial proof of the hockey stick identity.

SOLUTION: Elements of X_a are uniquely determined by choosing k elements from $\{a + 1, a + 2, \dots, n + 1\}$. We have

$$|\{a + 1, a + 2, \dots, n + 1\}| = n + 1 - a,$$

so

$$|X_a| = \binom{n+1-a}{k}.$$

By part (ii) and the ACP, we deduce that

$$\binom{n+1}{k+1} = |X| = \sum_{a=1}^{n-k+1} \binom{n+1-a}{k}.$$

Observing the binomial coefficient terms are exactly

$$\binom{n}{k}, \binom{n-1}{k}, \binom{n-2}{k}, \dots, \binom{k}{k},$$

we see that this may be rewritten as

$$\binom{n+1}{k+1} = \sum_{\ell=k}^n \binom{\ell}{k},$$

as desired.

Challenge

How many ways are there to write a nonnegative integer m as a sum of r positive integer summands? (We decree that the order of the addends matters, so $3 + 1$ and $1 + 3$ are two different representations of 4 as a sum of 2 nonnegative integers.) Develop a conjecture and prove it.

SOLUTION: After playing around for a while (OK, maybe a long while...), one comes to the conclusion that $\binom{m-1}{r-1}$ gives the desired count. For instance, we can represent 5 as the sum of 3 positive integers as $3 + 1 + 1$, $1 + 3 + 1$, $1 + 1 + 3$, $2 + 2 + 1$, $2 + 1 + 2$, or $1 + 2 + 2$, and $6 = \binom{4}{2}$.

A nice argument for this is given by the Balls and Walls method.¹ Imagine that we have m balls in a row. In order to represent m as a sum of r positive integers, we can place $r - 1$ walls in the spaces between the balls, taking care to not place two or more walls in a single gap. For example, the sum $7 = 1 + 3 + 2 + 1$ is represented by

$$\bullet | \bullet \bullet \bullet | \bullet \bullet | \bullet .$$

There is clearly a bijection between such ball-wall configurations and the sums we are counting, and each ball-wall configuration is specified by choosing $r - 1$ spots to place walls amongst the $m - 1$ gaps between balls; this number is, of course, $\binom{m-1}{r-1}$.

Challenge

Challenge problems are optional and should only be attempted after completing the previous problems.

¹ *Née* Stars and Bars, but that's a little too militaristic for Reed in my opinion.

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Answer the variation of the previous challenge problem in which we allow *nonnegative* integer summands.

SOLUTION: We again work with balls and walls where, for instance, $4 + 0 + 1 + 2 + 0$ is represented by

$$\bullet \bullet \bullet \bullet \mid \mid \bullet \mid \bullet \bullet \mid.$$

There are m balls and $r - 1$ walls which can be placed arbitrarily in $m + r - 1$ slots. Once the walls are placed, the ball positions are fixed, so there are

$$\binom{m + r - 1}{r - 1}$$

such summations.²

² This number is also the number of *multisets* of cardinality m taken from a set of cardinality r . A multiset taken from a set A is a function $m: A \rightarrow \mathbb{N}$. We think of m as recording the *multiplicity* of each element, so we can more casually think of multisets as sets that allow repetition. The cardinality of a multiset $m: A \rightarrow \mathbb{N}$ is $\sum_{a \in A} m(a)$, the sum of the multiplicities of the elements. We turn summations into multisets via the assignment that, for instance, takes $4 + 0 + 1 + 2 + 0$ to the multiset $\{1, 1, 1, 1, 3, 4, 4\}$ on $[5]$. (We leave it to the reader to precisely define the bijection in general.)