PROBLEM 1. Define a function $f\colon \mathbb{N} \to \mathbb{Z}$ by the piecewise formula

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{-1-n}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Show that f is a bijection, preferably by finding a two-sided inverse to f.

PROBLEM 2. Consider the function $g\colon \mathbb{Z} \to \mathbb{Z}$ given by

$$g(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Determine whether or not g is injective, and whether or not g is surjective.

Suppose A and B are sets and that $f\colon A \to B$ is a function. We define new functions

$$\begin{aligned} f_*\colon 2^A &\longrightarrow 2^B \\ X &\longmapsto f_*(X) = \{f(x) \mid x \in X\} \end{aligned}$$

and

$$\begin{aligned} f^*\colon 2^B &\longrightarrow 2^A \\ Y &\longmapsto f^*(Y) = \{x \in A \mid f(x) \in Y\}. \end{aligned}$$

Many authors write $f(X)$ for $f_*(X)$ and $f^{-1}(Y)$ for $f^*(Y)$. This overloading of notation is harmless once one is used to images and preimages, but we have chosen a more precise notation for this first encounter.

We call $f_*(X)$ the *image of X along f*, and $f^*(Y)$ the *preimage of Y along f*.

PROBLEM 3. Create a function $f\colon \{a, b, c, d\} \to \{1, 2, 3, 4\}$, and consider some examples of images and preimages of subsets.

PROBLEM 4. Express surjectivity of f in terms of f_*. Separately, express surjectivity of f in terms of f^*. What about injectivity of f?

PROBLEM 5. For $f\colon A \to B$, $X_1, X_2 \in 2^A$, and $Y_1, Y_2 \in 2^B$.

(a) Prove that $f_*(X_1 \cap X_2) \subseteq f_*(X_1) \cap f_*(X_2)$.
(b) Give a simple example showing that equality does not necessarily hold in the statement you just proved.
(c) True or false:
    i. $f_*(X_1 \cup X_2) = f_*(X_1) \cup f_*(X_2)$
    ii. $f^*(Y_1 \cup Y_2) = f^*(Y_1) \cup f^*(Y_2)$
    iii. $f^*(Y_1 \cap Y_2) = f^*(Y_1) \cap f^*(Y_2)$.

Challenge

Challenge problems are optional and should only be attempted after completing the previous problems.

Show that for every function $f\colon A \to B$ and subsets $X \in 2^A$, $Y \in 2^B$, we have

$$f_*(X) \subseteq Y \qquad \text{if and only if} \qquad X \subseteq f^*(Y).$$

This is an example of an *adjunction* — something to keep an eye out for if you ever encounter *category theory*.