Math 112 Homework, Friday Week 1

Typeset the text below the line, exactly, using LaTeX and *following the advice given on the second page*:

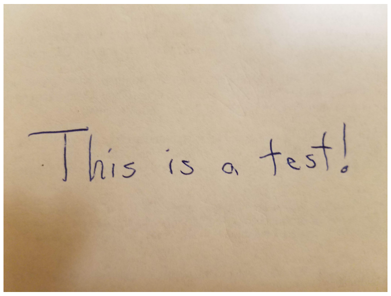
Let $a, b, c \in \mathbb{R}$ with $a \neq 0$. Then the most general solution to the equation $ax^2 + bx + c = 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Here is a derivation of a formula involving the difference of squares:

$$\begin{aligned}(x+y)(x-y) &= x(x-y) + y(x-y) \\ &= x^2 - xy + yx - y^2 \\ &= x^2 - y^2.\end{aligned}$$

Sometimes we will need to include a drawing like this:



We have the following containment of sets: $\emptyset \subset \{1, 2\} \subset \{1, 2, 3\}$.

» For the quadratic equation, you will need to use the LaTeX commands `\frac{}{}` and `\sqrt{}`. To get the $\pm$ sign, search online for a list of LaTeX mathematical symbols to find the correct code. (The answer to almost any LaTeX question is easily answered with a web search.)

» The symbol for the real numbers, $\mathbb{R}$, comes from the code `\mathbb{R}`. Since you are using it so much, you will want to make sure that

```
\newcommand{\R}{\mathbb{R}}
```

appears in the preamble of your document so that you can, instead, just type `\R`.

» Note how the quadratic equation is inline whereas its solution is displayed. (Here "display" means that that the solution appears on a separate, centered, line.) To display mathematics, enclose your code in `\[` and `\]` instead of `$`-signs.

» For the displayed difference of squares calculation, do not use `\[` and `\]`. Instead, use the construction

```
\begin{align*}
   blah &= blah\\
        &= blah\\
        &= etc.
\end{align*}
```

Equations are aligned at the `&` and a newline is produced by the `\\`. (You can control the space between the lines with something like this `\\[5pt]`, instead, to add 5 points of space.)

» You will sometimes want to insert complicated drawings in your files. To do this, you can create the drawing by hand and take a picture or with external software and then include the resulting graphics file. The code I used to get the result on the previous page was

```
\begin{center}
  \includegraphics[height=1.5in]{graphics/test.jpg}
\end{center}
```

(I kept the file `test.jpg` in a folder called "graphics".) To create complicated drawings in LaTeX, itself, read up on "tikz".

» To get the signs `{` and `}`, use the code `\{` and `\}`.

Math 112 Homework, Friday Week 2

Make sure to review the *Homework* portion of our Course Information sheet before writing up your solutions! For instance: **you will only receive full credit if you provide full explanations**. Also, your **solutions should consist *solely* of complete sentences**. Simply providing the correct numerical solution does not suffice. See the *Mathematical Writing* handout.

PROBLEM 1. Compute $\prod_{k=2}^{4}\left(1-\frac{1}{k^2}\right)$.

PROBLEM 2. Use induction to prove that

$$\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right)=\frac{n+1}{2n}.$$

for all $n \geq 2$.

PROBLEM 3. Let $A=\{1,3,5\}$, $B=\{4,5,6\}$, and $C=\{2,4,6,8\}$. Find the following sets:

(a) $(A \setminus B) \cup (B \setminus A)$.
(b) $C \setminus (B \setminus A)$.
(c) $C \cup (B \setminus A)$.
(d) $(C \cap A) \cup (C \cap B)$.
(e) $\{A\} \cap \{B\}$.

In this problem, your solution to (a) can take form "$(A\setminus B)\cup(B\setminus A) = \{\text{your answer here}\}$.", for example, and similarly for the other parts.

PROBLEM 4. Let X and Y be sets. Following the template given in class (see the video lecture or the end of our compiled lecture notes), prove that

$$(X \setminus Y) \cup (Y \setminus X) = (X \cup Y) \setminus (X \cap Y).$$

(Hint: you might want to split both parts of the proof into cases.)

PROBLEM 5. Let A, B, C, D be sets. Either prove the following or give an explicit counterexample showing that equality does not hold:

$$(A \cap C) \times (B \cap D) = (A \times B) \cap (C \times D).$$

PROBLEM 1. Given a set A, the *power set* of A is the set $\mathcal{P}(A)$ of all subsets of A:

$$\mathcal{P}(A) = \{B\colon B \subseteq A\}.$$

Another notation for the power set is 2^A. If A has n elements, it turns out that $\mathcal{P}(A)$ has 2^n elements (to make a subset, go to each element of A and make one of two choices: the element is in the subset, or it is not).

(a) List the elements of $\mathcal{P}(A)$ in the case where $A = \{x, y, z\}$.
(b) For a *general* set A (not necessarily the set A in part (a)) is the subset relation $\subseteq$ on $\mathcal{P}(A)$ reflexive? Is it symmetric? Is it transitive? For each of these properties, either give a proof that it holds or provide a (simple, concrete) counterexample.
(c) Is the subset relation $\subseteq$ on $\mathcal{P}(A)$ an equivalence relation?

PROBLEM 2. Let $X = \{x, y, z\}$, and consider the relation

$$R = \{(x, y), (y, y), (y, z)\} \subset X \times X$$

(a) List the elements of the smallest possible equivalence relation on X that contains R, explaining why the elements you add to R are required.
(b) How many equivalence classes does the resulting equivalence relation have?

The next three problems all refer to the same relation, which we now describe. Consider the set $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ of ordered pairs of integers with non-zero second component. Define an equivalence relation $\sim$ on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ by $(a, b) \sim (a', b')$ if $ab' = ba'$. For instance $(2, 6) \sim (1, 3)$ since $2 \cdot 3 = 6 = 6 \cdot 1$. Problems 3, 4, and 5, below, all refer to this equivalence relation.

PROBLEM 3. Prove that $\sim$ is an equivalence relation. (Follow the template, and make sure in your proof that you do not potentially divide by 0.)

PROBLEM 4. For each of the following equivalence classes, give a general description of all of the elements in the equivalence class and then give five concrete examples of elements in the equivalence class: $[(0, 2)]$, $[(3, 3)]$, and $[(3, 5)]$.

PROBLEM 5. The rational numbers are the set $\mathbb{Q} = \left\{\frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \neq 0\right\}$, i.e., of ordinary fractions.

(a) Describe a one-to-one correspondence (a bijection) between the rational numbers and the set of equivalence classes for $\sim$. (The equivalence classes are all infinite. Hence, each fraction is going to correspond to an infinite list of pairs of integers under this correspondence.)
(b) Find integers a and b such that the fractions $6/15$ and $2/5$ both correspond to the equivalence class $[(a, b)]$ under your correspondence. (No explantion necessary.)

PROBLEM 6. Show that the function

$$\begin{aligned} f\colon \mathbb{R} &\mapsto \mathbb{R} \\ x &\mapsto 2x + 5 \end{aligned}$$

is a bijection. (Do *not* do this by providing an inverse function. Instead, use our template: first prove it is injective, then prove it is surjective).

Problem 1. Recall that the *identity function* on a set A is the function $\mathrm{id}_A\colon A \to A$ defined by $\mathrm{id}_A(a) = a$ for all $a \in A$. In the reading, it was mentioned that a function $f\colon A \to B$ is a bijective if and only if there is a function $g\colon B \to A$ such that both $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$. (In this case, g is the inverse of f.) The point of this problem is to see that both of these conditions are necessary.

(a) Suppose $f\colon A \to B$ and $g\colon B \to A$ are functions.

   i. Show that if $g \circ f = \mathrm{id}_A$, then f is injective. (In this case, g is called a *left inverse* for f.

   ii. Show that if $f \circ g = \mathrm{id}_B$, then f is surjective. (In this case, g is called a *right inverse* for f.)

(b) Let $A = \{1, 2, 3\}$ and $B = \{w, x, y, z\}$. Define two functions $f\colon A \to B$ and $g\colon B \to A$ such that $g \circ f = \mathrm{id}_A$ and f is not a bijection.

Problem 2. Let $f\colon A \to B$ be a function between sets A and B. Let X and Y be subsets of B. Prove that

$$f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y).$$

[Notes: (i) follow the template for proving two sets are equal; (ii) recall the definition of the inverse image of a set: if $Z \subseteq B$, then $a \in f^{-1}(Z)$ means that $f(a) \in Z$.]

Problem 3.

(a) Let $a \in \mathbb{Z}$ have digits $a_k a_{k-1} \dots a_1 a_0$. In other words,

$$a = a_0 + a_1 \cdot 10 + a_2 \cdot 10^2 + \cdots + a_k \cdot 10^k.$$

For example, $547 = 7 + 4 \cdot 10 + 5 \cdot 10^2$. Show that, in general,

$$a = a_0 + a_1 + \cdots + a_k \bmod 9.$$

(Thus, for instance, $547 = 5 + 4 + 7 = 0 + 7 = 7 \bmod 9$.)

(b) Show how the above observation allows you to easily check that the integer 12345678 is divisible by 9.

The observation in the previous problem is the basis of the common and useful way of checking arithmetic called "casting out 9s". To check that your arithmetic is correct in adding a collection of multi-digit integers, first add all the digits, casting out 9s as you go to keep the sum small. The result is the sum of the digits modulo 9. Next add the digits of your answer, again casting out 9s. If your two results don't agree modulo 9, then you made an arithmetic mistake somewhere. For instance, consider the following calculation:

$$\begin{array}{rr} & 59284 \\ + & 27968 \\ \hline & 86252 \end{array}$$

Working modulo 9,

$$\begin{aligned} 59284 &= 5 + 9 + 2 + 8 + 4 = (5 + 4) + 9 + 2 + 8 \\ &= 0 + 0 + 2 + (-1) = 1 \end{aligned}$$

$$27968 = 2+7+9+6+8 = (2+7)+9+6+(-1) = 0+0+6-1 = 5.$$

Therefore,

$$59284 + 27968 = 1 + 5 = 6 \bmod 9.$$

On the other hand, again modulo 9,

$$86252 = 8+6+2+5+2 = -1+6+(2+5+2) = -1+6+9 = -1+6 = 5.$$

So

$$59284 + 27968 = 6 \neq 5 = 86252 \bmod 9,$$

which shows the arithmetic is faulty.

The digits of 59284 and 27968 were processed separately above, but they could have been combined: $5+9+2+8+4+2+7+9+6+8 =$ etc. mod 9, looking for pairs adding to 9 to discard.

Problem 4.

(a) Apply the method of casting out 9s to show the following arithmetic is mistaken. (As you go, look for digits that sum to 9 casting these out since that don't effect the sum modulo 9.)

$$\begin{array}{rr} & 183 \\ & 247 \\ & 346 \\ & 739 \\ + & 435 \\ \hline & 1960 \end{array}$$

What is the sum of the numbers modulo 9 (above the line), and what is the (incorrect) bottom-line sum modulo 9?

(b) Explain why the casting out 9s method of error-checking is not foolproof. Give a concrete example.

Problem 1. Let F be a field, and let $x, y \in F$. Prove that

$$(-x)y = -(xy).$$

Your proof should not use -1. Instead, use the *definition of an additive inverse*. Justify all steps in your proof. (You may use that $a \cdot 0 = 0$ for all $a \in F$ since we proved that earlier.) Here is a template:

> We need to show that $\boxed{}$ is the additive inverse of $\boxed{}$. We check this as follows:

Problem 2. Let F be a field, and let $x, y, z \in F$. Use the field axioms to show that if $x \neq 0$, then

$$xy = xz \quad \Longrightarrow \quad y = z.$$

Justify each step our your argument.

Problem 3. If x is an element of an ordered field, we define $x^1 := x$, and for each $n \geq 1$, we define $x^{n+1} := x \cdot x^n$.

Let F be an ordered field, and let $x \in F$ satisfy $x > 1$. Prove that $x^n > 1$ for all integers $n \geq 1$, being careful to cite relevant order axioms. (In our notes, we showed that $1 > 0$ in any ordered field. Since $x > 1$ and $1 > 0$, it follows by transitivity that $x > 0$. The fact that $x > 0$ will be relevant at some point in your proof. Also, since we have a recursive definition of x^n, a proof by induction would be natural.)

Math 112 Homework, Friday Week 6

PROBLEM 1. Fill in the following table, writing DNE if the quantity does not exist:

	sup	max	inf	min
$(-4, \pi)$				
$[6, 12)$				
$(-\infty, 3]$				
$\{1 + (-1)^n : n \in \mathbb{N}\}$				
$\{\sin(1/x) : x \in \mathbb{R}_{>0}\}$				

No explanation is required.

PROBLEM 2. Let S be a subset of an ordered field F, and suppose $\sup(S)$ exists. Fix $c \in F$, and define the set

$$c + S := \{c + s : s \in S\}.$$

Prove that $\sup(c + S) = c + \sup(S)$ by filling in the blanks in the template below. (Your write-up should include the full proof.)

Proof. We first prove that $c+\sup(S)$ is an upper bound for $c+S$. Let $x \in c+S$. The $x = c+s$ for some $s \in S$. Since $s \in S$, $\boxed{\qquad\qquad\qquad}$. From the additive translation axiom for ordered fields, $\boxed{\qquad\qquad\qquad}$. Hence, $c + \sup(S)$ is an upper bound for $c + S$.

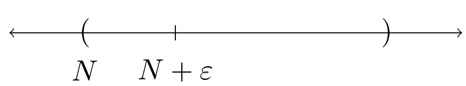
Next, we prove that $c+\sup(S)$ is the least upper bound for $c+S$. Let B be an upper bound for $c+S$. Then $B-c$ is an upper bound for S. To see this, let $s \in S$. Then $\boxed{\qquad\qquad\qquad}$

Since $B - c$ is an upper bound for S and $\sup(S)$ is the *least upper bound*, $\boxed{\qquad\qquad\qquad}$. It follows that $c + \sup(S) \leq B$, as desired. $\square$

PROBLEM 3. (We can approximate the infimum of a set arbitrarily closely with an element in the set.) Prove the following, imitating the proof of Proposition 4 in the reading for Friday, Week 5:

Let S be a subset of an ordered field F, and suppose that $N := \inf S$ exists. Given $\varepsilon \in F$ with $\varepsilon > 0$, there exists $s \in S$ such that $s - N < \varepsilon$.

Helpful picture. In trying to prove this result, the following picture (for the case $F = \mathbb{R}$ and S an interval) may help:



Note that (i) $s - N < \varepsilon$ is equivalent to $s < N + \varepsilon$ and (ii) since $N = \inf(S)$ and $s \in S$ it follows that s is in the interval $[N, N + \varepsilon)$.

PROBLEM 4. Prove that $\mathbb{C}$ satisfies the distributivity axiom for a field. (Work directly from our definition of $\mathbb{C}$ as $\mathbb{R}^2$ with a certain addition and multiplication, i.e., use the (a, b) notation as opposed the $a + bi$ notation.)

PROBLEM 5. Compute the following and express in the form $a + bi$ with $a, b \in \mathbb{R}$:

(a) $(2 - 7i)(1 + 2i) + (4 + i)$

(b) $\dfrac{1}{2 - i}$

(c) $\dfrac{1 + 3i}{3 + i}$

(d) $\left(-\dfrac{1}{2} + \dfrac{\sqrt{3}}{2} i\right)^3$

Math 112 Homework, Friday Week 7

Note: For homework from now on (unless specified), we assume all of the basic arithmetic for $\mathbb{R}$ and $\mathbb{C}$ and the obvious properties having to do with inequalities for $\mathbb{R}$ without having to justify them using the field and order axioms.

PROBLEM 1. Let $z, w \in \mathbb{C}$. Say $z = a + bi$ and $w = c + di$ with $a, b, c, d \in \mathbb{R}$. Prove that $\overline{zw} = \overline{z}\,\overline{w}$.

PROBLEM 2. Compute the following and express your answers in the form $a+bi$ with $a, b \in \mathbb{R}$:

(a) $\overline{4 - 8i}$.
(b) $|3 - 4i|$.
(c) $(1 - 2i)^2$.
(d) $\mathrm{Im}(2 + 5i + i(3 - 7i) + 17)$.
(e) $(4 + 3i)/(3 + 2i)$.

PROBLEM 3. Let F be an ordered field or the complex numbers. In class, we proved the triangle inequality:

$$|u + v| \leq |u| + |v| \tag{1}$$

for all $u, v \in F$. It turns out that easy substitutions for u and v yield the useful *reverse triangle inequality*:

$$|x - y| \geq \big|\, |x| - |y| \,\big|$$

for all $x, y \in F$.

We prove the reverse triangle inequality in two steps, first proving that $|x - y| \geq |x| - |y|$ and then proving that $|x - y| \geq |y| - |x|$ for all $x, y \in F$. The result then clearly follows. At no point in the following should you revert to using the definition of $|\ |$, which is, after all, defined differently for an ordered field and for $\mathbb{C}$.

(a) Find substitutions for u and v that transform the ordinary triangle inequality, (1), into the inequality $|x - y| \geq |x| - |y|$. (The substitutions will be simple expressions involving x and y. Hint: note that our objective is equivalent to $|x| \leq |x - y| + |y|$.)
(b) Use part (a) and the fact that $|-s| = |s|$ for all $s \in F$ to show $|x - y| \geq |y| - |x|$.

PROBLEM 4. Give the polar forms for the five solutions to $z^5 = 32$.

PROBLEM 5. Let $z = 8 + 9i$ to polar form. Use a calculator to compute the approximate value of $\arg(z)$ in *degrees*, and then use that to approximate the polar form for z.

PROBLEM 6. Let D be a nonempty subset of $\mathbb{R}$. Let $f\colon D \to \mathbb{R}$ and $g\colon D \to R$ be functions. Recall the notation

$$f(D) := \{f(x) : x \in D\} \quad \text{and} \quad g(D) := \{g(x) : x \in D\}.$$

Define $h\colon D \to \mathbb{R}$ by $h(x) := f(x) + g(x)$. (This function h is usually denoted $f + g$, for obvious reasons). Suppose that $f(D)$ and $g(D)$ are bounded above (so their suprema exist by completeness of $\mathbb{R}$).

(a) Show that $h(D)$ is bounded above by $\sup f(D) + \sup g(D)$. (Start: Let $y \in h(D)$. Therefore, $y = h(x)$ for some $x \in D$.)
(b) Since $h(D)$ is bounded above, it has a supremum by completeness of $\mathbb{R}$. Show that $\sup h(D) \leq \sup f(D) + \sup g(D)$.
(c) Find two specific functions $f, g : [-1, 1] \to \mathbb{R}$ such that we have a strict inequality $\sup h([-1, 1]) < \sup f([-1, 1]) + \sup g([-1, 1])$.

Math 112 Homework, Friday Week 9

PROBLEM 1. Find the limit of the following sequence, and provide an ε-N proof:

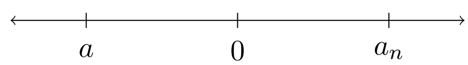
$$\left\{ \frac{3n^4 + 6n^2 + 1}{n^4 + 2n^2 + 3} \right\}.$$

PROBLEM 2. Find the limit of the following sequence, and provide an ε-N proof:

$$\left\{ \frac{(-1)^n}{n^2} \right\}.$$

PROBLEM 3. Suppose $\{a_n\}$ is a sequence of positive real numbers and that $\lim_{n\to\infty} a_n = a$.

(a) Use an ε-N argument to prove that $a \geq 0$. Hint: suppose that $a < 0$ and argue that this would imply $a_n < 0$ for some n. The picture below might help:



What should we take ε to be in order to force a_n to be be negative for large n (in contradiction to the fact that $a_n > 0$ is positive for all n)?

(b) Is it necessarily true that $a > 0$? (Give a proof or explicit counterexample.)

PROBLEM 4. Let $\{a_n\}$ be a sequence of complex numbers, and consider the sequence $\{|a_n|\}$ of real numbers.

(a) Suppose $\lim_{n\to\infty} a_n = a$. Prove that $\lim_{n\to\infty} |a_n| = |a|$ using an ε-N argument. (The reverse-triangle inequality may be of use.)

(b) Give an example for which $\lim_{n\to\infty} |a_n|$ exists but $\lim_{n\to\infty} a_n$ does not.

PROBLEM 5. Prove that $\lim_{n\to\infty} \frac{1}{n^2} \neq 3$ straight from the definition of the limit, i.e., using an ε-N argument. (You will need to find a specific $\varepsilon > 0$ for which there is no response N.)

PROBLEM 6. Find the limit of the following sequences, proving your results using our "new-from-old" limit theorem (not ε-N proofs). Use enough steps so that we can clearly see how that theorem is being used.

(a)

$$\left\{ \frac{4n^2 + 5}{8n^2 - 2n + 3} \right\}$$

(b)

$$\left\{ \frac{n}{n^2 + 3} \right\}.$$

PROBLEM 1. State whether each of the following is true or false for a real sequence. If true, give a justification. If false, give the simplest and most concrete counterexample you can think of. "Monotone" means either monotone increasing or monotone decreasing.

(a) If a bounded sequence is monotone, it's convergent.
(b) If a convergent sequence is monotone, it's bounded.
(c) If a convergent sequence is bounded, then it's monotone.
(d) If a sequence is bounded, then it's convergent.

PROBLEM 2. In order to evaluate the expression

$$\sqrt{3+\sqrt{3+\sqrt{3+\cdots}}},$$

define a real sequence by $a_1 = \sqrt{3}$ and $a_{n+1} = \sqrt{3+a_n}$, and then consider $\lim\limits_{n\to\infty} a_n$.

(a) Use induction to prove that the sequence in monotone increasing.
(b) Use induction to prove that the sequence is bounded above.
(c) By the monotone convergence theorem, we conclude that $\lim_{n\to\infty} a_n = a$ for some $a \in \mathbb{R}$. Proceed as in our notes to find the value of a.

PROBLEM 3. The point of this problem is to show that every convergent sequence is a Cauchy sequence (see the reading for Friday, Week 9). Let $\{a_n\}$ be a sequence complex numbers, and suppose that $\lim_{n\to\infty} a_n = a$ for some $a \in \mathbb{C}$. Use an $\varepsilon/2$-argument to prove that $\{a_n\}$ is a Cauchy sequence. (Pointers: Start by fixing $\varepsilon > 0$. Use the definition of $\lim_{n\to\infty} a_n = a$ but with $\varepsilon/2$. Next, look at the definition of a Cauchy sequence and use the triangle inequality. The idea is that all points in the sequence get close to a eventually and hence get close to each other. Note that $a_m - a_n = (a_m - a) - (a_n - a)$.)

PROBLEM 4. Let $\{a_n\}$ be a Cauchy sequence of complex numbers. Prove that $\{a_n\}$ is bounded, directly from the definition of a Cauchy sequence. (Idea: Let $\varepsilon = 1$ and use the fact that the sequence is Cauchy to get an N such that $m, n > N$ imply $|a_m - a_n| < \varepsilon = 1$. In particular, this implies that $|a_{N+1} - a_n| < 1$ for all $n > N$. In other words, a_n is in the ball of radius 1 centered at a_{N+1} whenever $n > N$. Now try to imitate the proof that convergent sequences are bounded given in our notes but using a_{N+1} in place of the limit of the sequence. You will need the reverse triangle inequality.)

PROBLEM 5. Use the geometric series test to determine whether each of the following series converges. If a series does converge, find its limit.

(a) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3}{8}\right)^n$.

(b) $\displaystyle\sum_{n=3}^{\infty} (-1)^n \frac{3^{n+2}}{10^n}$. Note that the sum starts at $n = 3$.

(c) $\displaystyle\sum_{n=0}^{\infty} \pi^n e^{-n}$.

(d) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{4} + \frac{\sqrt{3}}{4} i\right)^n$. Express your answer in the form $a + bi$.

First state whether why the sequence converges or diverges, and then find the sum, if it exists.

Math 112 Homework, Week 11

As usual, show your work and use sentences in order to receive credit.

PROBLEM 1. Use the comparison theorem (not the limit comparison theorem) to determine whether the following series converge. You may use the fact that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. (Show your work, as always.)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^3 + n^2 + 5}$ (b) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{\sqrt{n^2 - 5}}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n7^n}$

(d) $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt{n+1} - \sqrt{n-1}\right)$ (e) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n - e^{-n}}{5^n + e^{-n}}$

PROBLEM 2. What does the n-th term test say about the convergence or divergence of the following series?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2 + 5n - 7}{2n^2 + 3n - 6}$ (b) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{\sqrt{n^2 - 5}}$ (c) $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{\ln(n)}$.

PROBLEM 3. Sum the series:

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n-1)}.$$

Hint: partial fractions.

PROBLEM 4. Let $\{a_n\}$ be a sequence of nonnegative real numbers. Prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if its sequence of partial sums $\{s_n\}$ is bounded. (We have done this problem, but writing it up on your own will help in understanding the key role the monotone convergence theorem plays in series tests. Also: note that this is an "if and only if" proof. So you need to prove both implications.)

Note: In the following, you may use the fact that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. As always, *use sentences to explain your reasoning.*

PROBLEM 5. Use *the limit comparison test* to determine whether the following series converge. (Use limit comparison even if you know another method.)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)(n+3}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n+3}{3n^2 + 2n + 9}$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n3^n + 2}$.

PROBLEM 6. Are the following series absolutely convergent, conditionally convergent, or divergent?

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$ (b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n^3+2}$ (c) $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{e^n}{n^2}.$

PROBLEM 7. Apply the ratio test to each of the following series, and state what conclusion may be drawn:

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4}$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n2^n}$ (c) $\displaystyle\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n(n+1)!}$ (d) $\displaystyle\sum_{n=1}^{\infty}\frac{3^n n!}{(2n)!}.$

PROBLEM 8. Apply the integral test to each of the following series, and state what conclusion may be drawn:

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4}$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{n}{e^n}$ (c) $\displaystyle\sum_{n=1}^{\infty}\frac{3n^2+2}{n^3+2n+1}.$

PROBLEM 9.

(a) Why doesn't the integral test directly apply to determine convergence or divergence of the series

$$\sum_{n=1}^{\infty}\frac{\cos(n)}{n^3}\ ?$$

(b) What does the p-series test say about the convergence or divergence of the series

$$\sum_{n=1}^{\infty}\frac{1}{n^{1+e^{-n}}}\ ?$$

The group problems from Wednesday and Friday of Week 11 may be useful.

PROBLEM 1. Find $\lim_{x\to 3}(x^2+x+4)$ and provide an ε-δ proof. (Hint: at some point, you will need to factor a quadratic in order to find $|x-3|$.)

PROBLEM 2. We have encountered two notions for limits: one for sequences (using ε-N) and one for functions (using ε-δ). This problem explains the connection. Suppose that $\lim_{x\to a} f(x) = L$, and let $\{x_n\}$ be a sequence such that $\lim_{n\to\infty} x_n = a$ and such that no x_n equals a. Show that the sequence $\{f(x_n)\}$ converges to L. In your proof be sure to point out where the fact that no x_n equals a is used. (The converse of this result also holds but you are not asked to prove that here.) Advice: on scratch paper, write the ε-δ definition for $\lim_{x\to a} f(x) = L$ and the ε-N definitions for $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} f(x_n) = L$.

PROBLEM 3. Suppose that f and g are functions that are differentiable at some point a. Use the definition of the derivative, the definition of $f+g$, and our limit theorems for functions to prove that $f+g$ is differentiable at a and that $(f+g)'(a) = f'(a) + g'(a)$.

PROBLEM 4. Consider the function

$$\begin{aligned} f\colon (0,\infty) &\to \mathbb{R} \\ x &\mapsto \frac{1}{x^2}. \end{aligned}$$

Use the definition of the derivative and our limit theorems to prove that f is differentiable at every point $x \in (0,\infty)$ and to compute $f'(x)$.

PROBLEM 5. Compute the radius of convergence for each of the following series. (As always, show your work.)

$$\text{(a)} \sum_{n=1}^{\infty} \frac{(-1)^n}{n5^n} z^n \qquad \text{(b)} \sum_{n=0}^{\infty} \frac{n^3}{n!} z^n \qquad \text{(c)} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} z^n$$

PROBLEM 6. Consider an arbitrary polynomial in n of degree $d \geq 0$:

$$p(n) = c_d n^d + c_{d-1} n^{d-1} + \cdots + c_1 n + c_0,$$

where the c_i are constants and $c_d \neq 0$. What is the radius of convergence of $f(z) = \sum_{n=0}^{\infty} p(n) z^n$? (Feel free to use one of our ratio tests since once n is large, $p(n)$ will be nonzero.)

Problem 1. Let
$$g(x) = \sum_{n=0}^{\infty} \frac{1}{n6^n}(x-2)^n.$$
Describe the set of all points in $\mathbb{R}$ at which g converges (it will be an interval centered at 2). At which points does it converge absolutely, and at which does it converge conditionally? (Don't forget to check the endpoints of the interval of convergence.)

Problem 2. Let $\alpha \in \mathbb{R}$. The following notation will be useful: for each integer $k \geq 0$, let
$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}.$$
It might help to look up the binomial theorem (which will be the special case of this problem where α is an integer).

(a) Compute the Taylor series for $f(z) = (1+z)^\alpha$ centered at 0 (using the notation introduced above).
(b) What does your formula say in the case $\alpha = 4$? (In that case, the Taylor series will only have a finite number of nonzero terms.)
(c) What is the radius of convergence the Taylor series for f when $\alpha \notin \mathbb{Z}$? (When $\alpha \in \mathbb{Z}$, the Taylor series has only finitely many terms and always converges.)
(d) For each $n = 1, 2, 3, 4$, use the n-th order Taylor polynomial for f with $\alpha = 1/2$, to find fractions approximating $1/\sqrt{2}$ for $n = 1, \ldots, 4$. Use a computer to expand these fractions and $1/\sqrt{2}$ as decimals to see how well these approximations work.

Problem 3. Using our power series definitions of $\cos(z)$ and $\sin(z)$ for $z \in \mathbb{C}$, we have seen that $\cos'(z) = -\sin(z)$ and $\sin'(z) = \cos(z)$. Use these facts to prove that
$$\cos^2(z) + \sin^2(z) = 1$$
by completing the following steps. Let $f(z) := \cos^2(z) + \sin^2(z)$. Then: (i) using the chain rule, show that $f'(z) = 0$ and, thus, that f is a constant function, i.e. $f(z) = \alpha$ for some $\alpha \in \mathbb{C}$, and (ii) determine α by evaluating $f(z)$ at any convenient point.

Problem 4. (Exponentiation for complex numbers.) For this problem, fix a branch of the complex logarithm as we did in the group problems for Monday, Week 13. Thus, for $z \in \mathbb{C} \setminus \{0\}$, if $z = re^{i\theta}$ with $r \in \mathbb{R}_{>0}$ and $\theta \in (-\pi, \pi]$, then
$$\ln(z) = \ln(r) + i\theta.$$
Given $z, w \in \mathbb{C}$ with $z \neq 0$, define
$$z^w := e^{w\ln(z)}.$$
(Note that if $x, y \in \mathbb{R}$ with $x > 0$, this definition coincides with the usual one since $e^{y\ln(x)} = e^{\ln(x^y)} = x^y$.)

Compute the following (you will find they are both real numbers!):

(a) i^i
(b) $(-1)^i$.