PROBLEM 1. Compute $\sum_{k=-2}^{2}(3k+2)$ and show that it equals $3\sum_{k=-2}^{2} k + \sum_{k=-2}^{2} 2$.

SOLUTION: We have

$$\begin{aligned}\sum_{k=-2}^{2}(3k+2) &= (3(-2)+2)+(3(-1)+2)+(3(0)+2)+(3(1)+2)+(3(2)+2)\\ &= -4-1+2+5+8\\ &= 10.\end{aligned}$$

On the other hand,

$$\begin{aligned}3\sum_{k=-2}^{2} k + \sum_{k=-2}^{2} 2 &= 3(-2-1+0+1+2)+(2+2+2+2+2)\\ &= 3(0)+10\\ &= 10.\end{aligned}$$

PROBLEM 2. Use induction to prove that each $n \geq 1$,

$$1\cdot 2 + 2\cdot 3 + \cdots + n\cdot(n+1) = \frac{n(n+1)(n+2)}{3}.$$

SOLUTION: We will prove this by induction. The base case, $n = 1$, holds since

$$1\cdot 2 = \frac{1(1+1)(1+2)}{3}.$$

Now suppose that the result holds for some $n \geq 1$. It follows that

$$\begin{aligned}1\cdot 2 + 2\cdot 3 + \cdots + (n+1)(n+2) &= (1\cdot 2 + 2\cdot 3 + \cdots + n(n+1)) + (n+1)(n+2)\\ &= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) \qquad \text{(by the induction hypothesis)}\\ &= (n+1)(n+2)\left(\frac{n}{3}+1\right) \qquad \text{(factoring)}\\ &= \frac{(n+1)(n+2)(n+3)}{3}\\ &= \frac{(n+1)((n+1)+1)((n+1)+2)}{3}.\end{aligned}$$

The result then holds for the case $n+1$, as well. The result follows by induction.

Alternative solution. We will prove this by induction. The base case, $n = 1$, holds since

$$\sum_{k=1}^{1} k(k+1) = 1(1+1) = 2 = \frac{1(1+1)(1+2)}{3}.$$

Now suppose that the result holds for some $n \geq 1$. It follows that

$$\begin{aligned}\sum_{k=1}^{n+1} k(k+1) &= \sum_{k=1}^{n} k(k+1) + (n+1)(n+2)\\ &= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) \qquad \text{(by the induction hypothesis)}\\ &= (n+1)(n+2)\left(\frac{n}{3}+1\right) \qquad \text{(factoring)}\\ &= \frac{(n+1)(n+2)(n+3}{3}\\ &= \frac{(n+1)((n+1)+1)((n+1)+2)}{3}.\end{aligned}$$

The result then holds for the case $n+1$, as well. The result follows by induction. □

Problem 3. Let $a > -1$ be a real number. Use induction to show that for all integers $n \geq 0$,

$$(1+a)^n \geq 1 + na.$$

(Note: for any nonzero real number x, we have that $x^0 = 1$, by definition.)

Solution: We will prove this by induction. The base case, $n = 0$, holds since

$$(1+a)^0 = 1 \geq 1 = 1 + 0 \cdot a.$$

Suppose the result holds from some $n \geq 0$. Then

$$\begin{aligned}(1+a)^{n+1} &= (1+a)^n(1+a)\\ &\geq (1+na)(1+a) \qquad \text{(by the induction hypothesis}\\ &\qquad\qquad\qquad\qquad \text{and the fact that } 1+a>0)\\ &= 1 + (n+1)a + na^2\\ &\geq 1 + (n+1)a \qquad \text{(since } na^2 \geq 0).\end{aligned}$$

Thus, the result then holds for $n+1$, too. The result holds for all $n \geq 0$ by induction.

Problem 1. Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ and $C = \{3, 4, 5\}$ Find the following:

(a) $A \cup B \cup C$
(b) $A \cap B \cap C$
(c) $A \setminus B$
(d) $B \setminus A$
(e) $(A \cup B) \cap C$
(f) $(A \cap C) \cup (B \cap C)$
(g) $(A \cap B) \cup C$
(h) $(A \cup B) \cap (A \cup C)$.

Solution:

(a) $A \cup B \cup C = \{1, 2, 3, 4, 5\}$
(b) $A \cap B \cap C = \{3\}$
(c) $A \setminus B = \{1\}$
(d) $B \setminus A = \{4\}$
(e) $(A \cup B) \cap C = \{1, 2, 3, 4\} \cap \{3, 4, 5\} = \{3, 4\}$
(f) $(A \cap C) \cup (B \cap C) = \{3\} \cup \{3, 4\} = \{3, 4\}$
(g) $(A \cap B) \cup C = \{2, 3\} \cup \{3, 4, 5\} = \{2, 3, 4, 5\}$
(h) $(A \cup B) \cap (A \cup C) = \{1, 2, 3, 4\} \cap \{1, 2, 3, 4, 5\} = \{1, 2, 3, 4\}$.

Problem 2. Suppose that A, B, C are sets with $A \subseteq B \subseteq C$. Prove or disprove:

$$C \setminus B \subseteq C \setminus A.$$

Solution: Let $x \in C \setminus B$. Then $x \in C$ and $x \notin B$. Since $A \subseteq B$ and $x \notin B$, it follows that $x \notin A$. Thus, $x \in C$ and $x \notin A$. It follows that $x \in C \setminus A$. Therefore. $C \setminus B \subseteq C \setminus A$.

Problem 3. Let $A = \{1, \{3, 6, 9\}, \{\emptyset\}\}$.

(a) What are the elements of A?
(b) Is $6 \in A$?
(c) Is $\{1\} \subseteq A$?
(d) Is $\emptyset \subseteq A$?
(e) Is $\emptyset \in A$?

Solution:

(a) The set A has three elements: 1, $\{3, 6, 9\}$, and $\{\emptyset\}$.
(b) No, but 6 is an element of the element $\{3, 6, 9\}$ of A.
(c) Yes, every element of $\{1\}$ is an element of A.

(d) Yes, the empty set is a subset of every set. (The statement that every element of the empty set is a subset of A is *vacuously* true. For this statement to be false, the empty set would need to contain an element.)

(e) No: $\emptyset$ is not one of the three elements of A. On the other hand, it is true that $\{\emptyset\} \in A$.

PROBLEM 4. Describe the following intersection of open intervals of $\mathbb{R}$

$$\bigcap_{n=1}^{\infty}(-1/n, 1/n) = (-1,1) \cap (-1/2,1/2) \cap (-1/3,1/3) \cap (-1/4,1/4) \cap \cdots$$

as simply as possible. What are its elements? Do the same for

$$\bigcup_{n=1}^{\infty}(-1/n, 1/n) = (-1,1) \cup (-1/2,1/2) \cup (-1/3,1/3) \cup (-1/4,1/4) \cup \cdots .$$

SOLUTION: The only element that is $(-1/n, 1/n)$ for all integers $n \geq 1$ is 0. Therefore,

$$(-1,1) \cap (-1/2,1/2) \cap (-1/3,1/3) \cap (-1/4,1/4) \cap \cdots = \{0\}.$$

We have

$$(-1,1) \cup (-1/2,1/2) \cup (-1/3,1/3) \cup (-1/4,1/4) \cup \cdots = (-1,1).$$

Math 112 Group problems, Monday Week 2

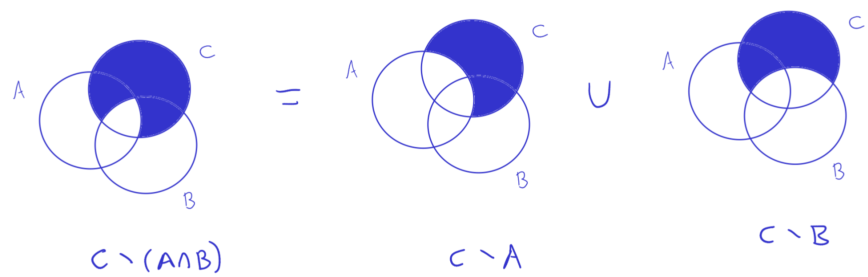
PROBLEM 1. **Proposition.** Let A, B, C be sets. Then

$$C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B).$$

(a) Draw a Venn diagram that shows the Proposition is reasonable.
(b) Prove the Proposition.

SOLUTION:

(a)



(b) Let $x \in C \setminus (A \cap B)$. Then $x \in C$ and $x \notin A \cap B$. Since $x \notin A \cap B$, it follows that x is not in both A and B. Thus, we have two cases to consider. First, say $x \notin A$. Then $x \in C$ and $x \notin A$. Thus, $x \in C \setminus A$, and, hence, $x \in (C \setminus A) \cup (C \setminus B)$, as desired. Second, if $x \notin B$, then $x \in C \setminus B$, and it again follows that $x \in (C \setminus A) \cup (C \setminus B)$.

Conversely, now assume that $x \in (C \setminus A) \cup (C \setminus B)$. Therefore, $x \in C \setminus A$ or $x \in C \setminus B$. Again, we have two cases. First, suppose $x \in C \setminus A$. Then $x \in C$ and $x \notin A$. It follows that $x \in C$ and $x \notin A \cap B$. Therefore, $x \in C \setminus (A \cap B)$. The second case follows similarly.

PROBLEM 2. Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Write all of the elements of $A \times B$.

SOLUTION: The elements of $A \times B$ are:

$$(1, a), (1, b), (1, c), (2, a), (2, b), (2, c).$$

PROBLEM 3. Let A, B, C be sets. Show that

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

SOLUTION:

Proof. Let $(x, y) \in A \times (B \cap C)$. Then $x \in A$ and $y \in B \cap C$. Since $y \in B \cap C$, it follows that $y \in B$ and $y \in C$. Since $x \in A$ and $y \in B$, it follows that $(x, y) \in A \times B$. Since $x \in A$ and $y \in C$, it follows that $(x, y) \in A \times C$. Therefore $x \in (A \times B) \cap (A \times C)$.

Conversely, suppose that $(x, y) \in (A \times B) \cap (A \times C)$. Then $(x, y) \in A \times B$ and $(x, y) \in A \times C$. We conclude that $x \in A$ and that y is in both B and C, i.e., $y \in B \cap C$. Hence, $(x, y) \in A \times (B \cap C)$. $\square$

PROBLEM 1. Let A be the set of all lines in the plane. Is the relation "is parallel to" on A an equivalence relation? If not, which properties prevent if from being so. (Take "parallel" to mean "same slope" rather than "non-intersecting". How does this affect your answer?)

SOLUTION: This is an equivalence relation but would not be if we took "parallel" to mean "non-intersecting".

Reflexivity. A line has the same slope as itself. Hence, the relation is reflexive. If, on the other hand, we took "parallel" to mean "non-intersecting", then reflexivity would not hold.

Symmetry. If line L is parallel to line M, then line M is parallel to line L. Hence, the relation is symmetric.

Transitivity. If L, M, N are lines and L is parallel to M and M is parallel to N, then L is parallel to N. Hence, the relation is transitive. If we took "parallel" to mean non-intersecting, we would not have transitivity: consider the case where $L = N$.

PROBLEM 2. Let A be the set of all lines in the plane. Is the relation "is perpendicular to" on A an equivalence relation? If not, which properties prevent if from being so.

SOLUTION: This is not an equivalence relation.

Reflexivty. A line is not perpendicular to itself.

Symmetry. If line L is perpendicular to line M, then M is perpendicular to L. Hence, the relation is symmetric.

Transitivity. Let L, M, N be lines. If L is perpendicular to M and M is perpendicular to N, then L and N are parallel, not perpendicular. Hence, the relation is not transitive.

PROBLEM 3. For $a, b \in \mathbb{Z}$, say $a \sim b$ if $a - b = 2k$ for some $k \in \mathbb{Z}$. In other words, $a \sim b$ if $a - b$ is an even integer. Prove that $\sim$ is an equivalence relation on $\mathbb{Z}$ following the template below:

Theorem. Define a relation $\sim$ on a set A by blah, blah, blah. Then $\sim$ is an equivalence relation.

Proof. Let $a, b, c \in A$.

Reflexivity. We have $a \sim a$ since blah, blah, blah. Therefore, $\sim$ is reflexive.

Symmetry. Suppose that $a \sim b$. Then, blah, blah, blah. It follows that $b \sim a$. Therefore $\sim$ is symmetric.

Transitivity. Suppose that $a \sim b$ and $b \sim c$. Then blah, blah, blah. It follows that $a \sim c$. Therefore, $\sim$ is transitive.

Since $\sim$ is reflexive, symmetric, and transitive, it follows that $\sim$ is an equivalence relation. $\square$

Solution:

Proof. Let $a, b, c \in \mathbb{Z}$.

Reflexivity. For each $a \in \mathbb{Z}$, we have $a \sim a$ since $a - a = 0 = 2 \cdot 0$, i.e., $a - a$ is even. Therefore, $\sim$ is reflexive.

Symmetry. Suppose that $a \sim b$. Then, $a - b = 2k$ for some $k \in \mathbb{Z}$. But then, $b - a = 2(-k)$. It follows that $b \sim a$. Therefore $\sim$ is symmetric.

Transitivity. Suppose that $a \sim b$ and $b \sim c$. Then $a - b = 2k$ and $b - c = 2k'$ for some $k, k' \in \mathbb{Z}$. But then

$$a - c = (a - b) + (b - c) = 2k - 2k' = 2(k - k').$$

It follows that $a \sim c$. Therefore, $\sim$ is transitive.

Since $\sim$ is reflexive, symmetric, and transitive, it follows that $\sim$ is an equivalence relation.

$\square$

PROBLEM 1. Let $S = \{1, 2, 3, 4\}$.

(a) Partition S into two sets S_1 and S_2. Describe the corresponding equivalence relation formally as a subset of $S \times S$.

(b) Repeat, using different sets S_1, and S_2.

(c) What is the name for the equivalence $\sim$ relation on S whose equivalence classes are

$$\{1\}\,,\ \{2\}\,,\ \{3\}\,,\ \text{and}\ \{4\}\,.$$

SOLUTION:

(a) Let $S_1 = \{1, 2\}$ and $S_2 = \{3, 4\}$. The corresponding equivalence relation is:

$$\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}\,.$$

(b) Let $S_1 = \{1, 2, 3\}$ and $S_2 = \{4\}$. The corresponding equivalence relation is:

$$\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 4)\}\,.$$

(c) This equivalence relation is just *equality*, $=$.

PROBLEM 2. Let $n \in \mathbb{Z}$ and consider the equivalence relation $\sim$ on $\mathbb{Z}$ that defines $\mathbb{Z}/n\mathbb{Z}$, i.e., if $a, b \in \mathbb{Z}$, then $a \sim b$ if $a - b = nk$ for some $k \in \mathbb{Z}$. Prove that

$$a \sim a' \text{ and } b \sim b' \quad \Rightarrow \quad a + b \sim a' + b'.$$

SOLUTION:

Proof. Since $a \sim a'$ and $b \sim b'$, there exist $k, \ell \in \mathbb{Z}$ such that

$$a - a' = kn \ \text{ and } \ b - b' = \ell n.$$

It follows that

$$(a + b) - (a' + b') = (a - a') + (b - b') = kn + \ell n = (k + \ell)n.$$

Letting $m := k + \ell \in \mathbb{Z}$, we see

$$(a + b) - (a' + b') = mn.$$

Hence, $a + b \sim a' + b'$. $\square$

PROBLEM 3. Again consider the equivalence relation $\sim$ defining $\mathbb{Z}/n\mathbb{Z}$. This time, show

$$a \sim a' \text{ and } b \sim b' \quad \Rightarrow \quad ab \sim a'b'.$$

SOLUTION:

Proof. Since $a \sim a'$ and $b \sim b'$, there exist $k, \ell \in \mathbb{Z}$ such that

$$a - a' = kn \quad \text{and} \quad b - b' = \ell n.$$

It follows that

$$\begin{aligned} ab &= (a' + kn)(b' + \ell n) \\ &= a'b' + a'\ell n + kb'n + k\ell n^2 \\ &= a'b' + (a'\ell + kb' + \ell n)n. \end{aligned}$$

Letting $m := a'\ell + kb' + \ell n \in \mathbb{Z}$, we have

$$ab - a'b' = mn,$$

and, hence, $ab \sim a'b'$. $\square$

PROBLEM 1. Let $A := \{1, 2, 3, 4\}$ and $B := \{a, b, c\}$. Define $f: A \to B$ by $f(1) = f(3) = a$, $f(2) = b$, and $f(4) = c$.

(a) What are the domain and codomain of f?

(b) What is the formal definition of f as a relation (a subset of $A \times B$)?

(c) Is f injective? surjective? bijective?

SOLUTION:

(a) The domain is A and the codomain is B.

(b) Formally, f is defined by its graph:

$$\{(1, a), (2, b), (3, a), (4, c)\}.$$

(c) Since $f(1) = f(3)$, the function is not injective. However, f is surjective since its image is B: 1 is in the pre-image of a (as is 3), 2 is the pre-image of b and 4 is the pre-image of c. Since f is not injective, it is not a bijection.

PROBLEM 2. Consider the absolute value function:

$$\begin{aligned} f: \mathbb{R} &\to \mathbb{R} \\ x &\mapsto |x|. \end{aligned}$$

(a) Draw the graph of f.

(b) What is $\mathrm{im}(f)$, the image of f?

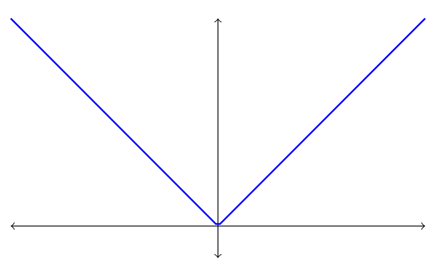
(c) Is f injective? (Prove or provide a concrete counterexample.)

(d) Is f surjective? (Prove or provide a concrete counterexample.)

(e) How are the answers to the last two questions reflected in your drawing of the graph of f?

SOLUTION:

(a)



Graph of f.

(b) The image of f is $\mathbb{R}_{\geq 0}$.

(c) The function f is not injective since, for instance $f(1) = f(-1)$.

(d) The function f is not surjective since $\operatorname{im}(f) = \mathbb{R}_{\geq 0} \subsetneq \operatorname{codomain}(f) = \mathbb{R}$. For instance, -1 is not in the image of f.

(e) We can see from the graph that f is not injective since there are some horizontal lines that meet the graph in two points.

(f) We can see from the graph that f is not surjective since there are some horizontal lines that meet the graph in no points.

Problem 3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x - 7$. Prove that f is bijective. (Follow the template.)

Solution:

Proof. To see that f is injective, let $x, y \in \mathbb{R}$ and suppose that $f(x) = f(y)$. It follows that $3x - 7 = 3y - 7$. Adding 7 to both sides of this equation gives $3x = 3y$. Then dividing by 3 gives $x = y$. Hence, f is injective.

To see that f is surjective, let $z \in \mathbb{R}$ (i.e. fix an arbitrary point in the codomain of f). We need to find $x \in \mathbb{R}$ (in the domain of f) such that $f(x) = z$. In other words, we need to solve the equation

$$3x - 7 = z$$

for x. We find $x = (z + 7)/3$. Check:

$$f((z + 7)/3) = 3(z + 7)/3 - 7 = z + 7 - 7 = z,$$

as required. Hence, f is surjective.

Since f is injective and surjective, it is a bijection. $\square$

PROBLEM 1. Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$.

(a) Give an example of two subsets X and Y of A and a function $f : A \to B$ such that $f(X \cap Y) \neq f(X) \cap f(Y)$.

(b) Is it possible to find subsets X and Y of A such that $f(X \cap Y) \not\subseteq f(X) \cap f(Y)$?

SOLUTION:

(a) For one example, let $f(1) = f(2) = a$ and $f(3) = b$, and let $X = \{1\}$ and $Y = \{2\}$. Then $f(X \cap Y) = f(\emptyset) = \emptyset$, while

$$f(X) \cap f(Y) = \{a\} \cap \{a\} = \{a\}\,.$$

(b) This is not possible. We proved that $f(X \cap Y) \subseteq f(X) \cap f(Y)$ for all functions f and subsets X, Y of the domain of f in the lecture.

PROBLEM 2. For each of the following functions, state why there is no inverse, or describe the inverse function.

(a)

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto |x|. \end{aligned}$$

(b)

$$\begin{aligned} g : \mathbb{R}_{\geq 0} &\to \mathbb{R} \\ x &\mapsto |x|. \end{aligned}$$

(c)

$$\begin{aligned} h : \mathbb{R}_{\geq 0} &\to \mathbb{R}_{\geq 0} \\ x &\mapsto |x|. \end{aligned}$$

SOLUTION:

(a) This function has no inverse since it is not bijective. It's not surjective since, for example, $-1 \notin \operatorname{im}(f)$. It's not injective since, for example, $|-1| = |1|$.

(b) This function has no inverse since it is not bijective. It's not surjective since, for example, $-1 \notin \operatorname{im}(g)$.

(c) The inverse of this function is

$$\begin{aligned} h^{-1} : \mathbb{R}_{\geq 0} &\to \mathbb{R}_{\geq 0} \\ x &\mapsto x. \end{aligned}$$

PROBLEM 3. Show that the function

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto 3x + 1 \end{aligned}$$

is a bijection by providing its inverse function. (Demonstrate that the function you produce is actually the inverse of f. You need to check both possible compositions are the identity.) [This is the second method of proving that a function is bijective. The first, which more closely follows the definition of bijectivity is to prove that the function is both injective and surjective.]

SOLUTION: The inverse function is

$$\begin{aligned} g\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto (x-1)/3. \end{aligned}$$

To verify that g is the inverse, we check

$$(f \circ g)(x) = f(g(x)) = f((x-1)/3) = 3\,((x-1)/3) + 1 = (x-1) + 1 = x,$$

and

$$(g \circ f)(x) = g(f(x)) = g(3x+1) = ((3x+1)-1)\,/3 = 3x/3 = x.$$

PROBLEM 4. Consider the functions $f(x) = x+1$ and $g(x) = 3x$, both with domain and codomain $\mathbb{R}$. Compute the following: (i) $g \circ f$, (ii) $(g \circ f)^{-1}$, (iii) f^{-1}, (iv) g^{-1}, and (v) $f^{-1} \circ g^{-1}$. Verify that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

SOLUTION:

(i) We have
$$(g \circ f)(x) = g(f(x)) = g(x+1) = 3(x+1) = 3x+3.$$

(ii) To find $(g\circ f)^{-1}$, we set $y = 3x+3$ and solve for x. We find $x = (y-3)/3$. Therefore, $(g\circ f)^{-1}(x) = (x-3)/3$ (with domain and codomain equal to $\mathbb{R}$).

(iii), (iv) Similarly, we find
$$f^{-1}(x) = x - 1 \quad \text{and} \quad g^{-1}(x) = x/3.$$

(v) Therefore,
$$(f^{-1} \circ g^{-1})(x) = f^{-1}(x/3) = x/3 - 1 = (x-3)/3 = (g \circ f)^{-1}.$$

Problem 1. Fill in the following addition and multiplication tables (using standard representatives for equivalence classes for convenience, e.g, 3 instead of [3]).

$\mathbb{Z}/5\mathbb{Z}$

$+$	0	1	2	3	4
0					
1					
2					
3					
4					

$\cdot$	0	1	2	3	4
0					
1					
2					
3					
4					

$\mathbb{Z}/6\mathbb{Z}$

$+$	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						

$\cdot$	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						

SOLUTION:

$\mathbb{Z}/5\mathbb{Z}$

$+$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

$\mathbb{Z}/6\mathbb{Z}$

$+$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	5	2	0	4	2
5	0	5	4	3	2	1

PROBLEM 2. Why are all of the tables in the previous problem symmetric about the diagonal from top-left to bottom-right? Do you see any other patterns?

SOLUTION: That is because addition and multiplication in $\mathbb{Z}/n\mathbb{Z}$ are commutative: $[a]+[b] = [b]+[a]$ and $[a][b] = [b][a]$. Said another way, $a + b = b + a \bmod n$ and $ab = ba \bmod n$.

Some other patterns:

(a) The rows of the addition table are cyclic shifts of each other.

(b) The equivalence classes for 0 and 1 behave as one might expect with respect to addition and multiplication.

(c) In the bottom row and last column of the multiplication tables, the classes besides $[0]$ are listed in reverse order.

PROBLEM 3. Let $a, b \in \mathbb{Z}$. When is $a = b \bmod 2$? When is $a = b \bmod 1$? When is $a = b \bmod 0$? List the equivalence classes in each case, i.e., the elements of $\mathbb{Z}/n\mathbb{Z}$, for $n = 2, 1, 0$.

SOLUTION: We have $a = b \bmod 2$ exactly when a and b are both even or if they are both odd. There are two equivalence classes, $[0]$ and $[1]$.

We have $a = b \bmod 1$ when $a - b = 1 \cdot k$ for some $k \in Z$. But there is always such a k, namely, $k = a - b$. Thus, $a = b \bmod 1$ holds for all a and b. There is one equivalence class, $[0]$.

We have $a = b \bmod 0$ when $a - b = 0 \cdot k$, i.e., when $a - b = 0$. So a and b are equal modulo 0 if and only if $a = b$. The equivalence classes are $[a]$ for $a \in \mathbb{Z}$, and each equivalence class just contains one element.

PROBLEM 4. Use modular arithmetic to find the last two digits of the following two numbers:

$$101^{(10^{1000}+2021)} \quad \text{and} \quad 99^{(10^{1000}+2021)}.$$

SOLUTION: To find the last two digits, we find standard representatives modulo 100. Now $101 = 1 \bmod 100$. So

$$101^{(10^{1000}+2021)} = 1^{(10^{1000}+2021)} = 1 \bmod 100.$$

The last two digits are 01. Similarly, $99 = -1 \bmod 100$. So

$$99^{(10^{1000}+2021)} = (-1)^{(10^{1000}+2021)} = -1 = 99 \bmod 100,$$

since $10^{10^{1000}+2021}$ is odd. The last two digits in this case are 99.

PROBLEM 5 (Challenge). Let $a_1 = 3$, and for $n > 0$, define $a_n = 3^{a_{n-1}}$. Thus, $a_2 = 3^3 = 27$, and $a_3 = 3^{3^3} = 3^{27}$. What is the last digit of a_{100}? (Hint: start by considering the last digits of 3, 3^2, 3^3, 3^4, etc., until you see a pattern. You may start to think that the number 4 is significant.)

SOLUTION: To find the last digit, work modulo 10. We have $3^4 = 81 = 1 \mod 10$. Therefore, the last digit of 3^n for any n is determined by the equivalence class of n modulo 4. For instance,

$$3^{n+4} = 3^n \cdot 3^4 = 3^n \bmod 10.$$

So to find a_{100}, we need to find a_{99} modulo 4. Since $3 = -1 \bmod 4$, we have

$$a_{99} = 3^{a_{98}} = (-1)^{a_{98}} = -1 = 3 \bmod 4$$

since a_{98} is odd. Then

$$a_{100} = 3^{a_{99}} = 3^3 = 27 = 7 \bmod 10.$$

If A and B are sets, say $A \sim B$ if there exists a bijection $A \to B$. All of the problems below refer to this relation

PROBLEM 1. Prove that $\sim$ is an equivalence relation.

Proof. Let A, B, C be sets.

Reflexivity. We have $A \sim A$ since the identity mapping $\mathrm{id}_A \colon A \to A$ defined by $\mathrm{id}_A(a) = a$ for all $a \in A$ is a bijection.

Symmetry. Suppose $A \sim B$. Then there exists a bijection $f \colon A \to B$. Since f is bijective, it has an inverse $f^{-1} \colon B \to A$, and that inverse is a bijection. Hence, $B \sim A$.

Transitivity. Suppose $A \sim B$ and $B \sim C$. Then there are bijections $f \colon A \to B$ and $g \colon B \to C$. Since the composition of bijections is a bijection, $g \circ f \colon A \to C$ is a bijection. Hence, $A \sim C$. □

PROBLEM 2. Let A, B be sets. If $A \sim B$, we say that A and B have the *same cardinality* and write $|A| = |B|$.

(a) If A is a finite set, describe all of the sets B in the equivalence class for A.

(b) Let $2\mathbb{Z}$ denote the set of even integers. Prove that $\mathbb{Z}$ and $2\mathbb{Z}$ have the same cardinality.

Solution.

(a) The equivalence class for A consist of all sets that have the same number of elements as A.

(b) The mapping

$$\begin{aligned} \mathbb{Z} &\to 2\mathbb{Z} \\ n &\to 2n \end{aligned}$$

is a bijection.

□

PROBLEM 3. A set having the same cardinality as $\mathbb{N}$ is said to be *countably infinite*. To say that a set X is countably infinite means the its elements may be listed in a line:

$$x_0, x_1, x_2, \dots$$

Given this list, we get a bijection $f \colon \mathbb{N} \to X$ by letting $f(n) := x_n$. Conversely, given a bijection $f \colon \mathbb{N} \to X$, we create the list as

$$f(0), f(1), f(2), \dots$$

Prove that $\mathbb{Z}$ is countably infinite.

Solution. Here is a list of the elements of $\mathbb{Z}$:

$$0, -1, 1, -2, 2, -3, 3, \ldots$$

$\square$

Problem 4. Is $\mathbb{Q}$, the set of rational numbers, countably infinite? If so, describe your list of the rationals. Please consider this question for a while before proceeding to the next problem (which provides a solution). If you have prior knowledge regarding this question, please don't give away the solution to your fellow group members.

Problem 5. Fill in the following table so that the entry in its a-th column and b-th row is the reduced version of the fraction a/b:

	1	2	3	4	5	6	7
1	1	2					
2	$\frac{1}{2}$						
3							
4							
5							
6							
7							

Create a list of the positive rational numbers as follows: Follow the snake path starting in the upper-left corner of the box. Each time you reach a fraction, if that number is not in your list already, add it to your list. Thus, your list will start: $1, 2, 1/2, 1/3, 3, \ldots$. Continue until you get to the seventh diagonal. Imagine that the table extends infinitely in both directions so that you may continue the list indefinitely. Does your list then contain every positive rational number exactly once? What does this say about the cardinality of $\mathbb{Q}_{>0}$?

Solution.

	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$
3	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{5}{3}$	2	$\frac{7}{3}$
4	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{7}{4}$
5	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1	$\frac{6}{5}$	$\frac{7}{5}$
6	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1	$\frac{7}{6}$
7	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$\frac{1}{7}$	1

$$1, 2, \frac{1}{2}, \frac{1}{3}, 3, 4, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{1}{5}, 5, 6, \frac{5}{2}, \frac{4}{3}, \frac{3}{4}, \frac{2}{5}, \frac{1}{6}, \frac{1}{7}, \frac{3}{5}, \frac{5}{3}, 7.$$

This extended list contains every element of $\mathbb{Q}_{>0}$ exactly once. Hence $\mathbb{Q}$ is countably infinite, and $|\mathbb{N}| = |\mathbb{Q}_{>0}|$. $\square$

PROBLEM 6.

(a) Suppose sets X and Y are countably infinite. List the elements of X and Y:

$$X\colon\ x_0, x_1, x_2, \ldots$$
$$Y\colon\ y_0, y_1, y_2, \ldots$$

Show that $X \cup Y$ is countably infinite. (Thus, X and Y are "listable". Create a list for $X \cup Y$. At first, you might assume that X and Y are disjoint, and then consider how to modify your list if $X \cap Y \neq \emptyset$.)

(b) Above, we have shown that $\mathbb{Q}_{>0}$ is countably infinite. Argue that $\mathbb{Q}$ is countably infinite.

Solution.

(a) Create the list

$$x_0, y_0, x_1, y_1, x_2, y_2, \ldots$$

Next read the list from the start, removing any elements that are repeated (these will be exactly the elements of $X \cap Y$).

(b) Using what we just showed, we can list the elements of $\mathbb{Q}_{<0} \cup \mathbb{Q}_{0>0}$, and then prepend a 0. This gives a list of $\mathbb{Q} = \{0\} \cup \mathbb{Q}_{<0} \cup \mathbb{Q}_{>0}$.

Consider the equivalence relation on sets defined by $A \sim B$ if there exists a bijection from A to B. We say two sets have the *same cardinality* if they are equivalent under this equivalence relation, and we write $|A| = |B|$. Any set with the same cardinality as $\mathbb{N}$ is *countably infinite*. For a set A to be countably infinite means that its elements can be listed in an unending line, $a_0, a_1, a_2, \ldots$. (The resulting bijection $\mathbb{N} \to A$ sends n to a_n.) Last time, we showed that the rational numbers are countably infinite.

Problem 1. (Cantor's diagonal argument, 1891) It turns out that the real numbers are not countable, i.e., they cannot be put into bijection with the natural numbers. Here, we will give the slightly easier argument that the subset of the real decimals containing only 0s and 1s is not countable. Define *binary decimals* to be the real numbers of the form $0.a_1a_2a_3\ldots$ where each $a_i \in \{0, 1\}$. A binary decimal would look like $0.0110001010011\ldots$ For sake of contradiction, suppose you could list the binary decimals. Your list would then look something like this (leaving off the initial "0."):

n													
0	1	1	0	0	0	1	0	1	0	0	1	1	$\ldots$
1	0	0	0	1	0	1	0	0	1	0	0	0	$\ldots$
2	1	0	0	0	0	0	1	0	0	0	0	1	$\ldots$
3	0	1	0	0	1	1	0	1	1	1	1	0	$\ldots$
4	1	0	1	1	1	0	0	0	1	1	0	1	$\ldots$
5	0	1	1	1	1	1	0	0	0	1	1	1	$\ldots$
6	1	1	1	1	1	0	0	0	0	0	0	1	$\ldots$
7	1	0	1	0	0	0	0	0	0	0	0	1	$\ldots$
8	1	1	1	0	0	0	1	0	0	0	0	0	$\ldots$
9	0	1	1	0	1	0	0	1	0	1	1	0	$\ldots$
10	0	0	1	0	0	1	1	0	1	0	1	0	$\ldots$
11	0	1	0	1	0	1	0	0	0	0	0	1	$\ldots$
$\vdots$							$\vdots$						

We will show that your list is not complete. Read off the diagonal from the above table: $0.100011000111\ldots$. Except for the initial "0.", swap the 0s and 1s in this number: $0.011100111000\ldots$ Why isn't this number in the list? Next, place this number at the beginning of your list. Do you now have a complete list of the binary decimals?

Solution. It differs from the zero-th number in the list in its first decimal, from the first number in its second decimal, and so on. If we place the newly formed number at the beginning of the list, we can perform the same procedure, going down the diagonal, to produce a binary decimal that is not in this newly formed list. No matter what linear list of binary decimals we create, it will not contain all of the binary decimals.

Problem 2. If A and B are sets, we write $|A| < |B|$ if there exists an injection $A \to B$ but there exists no bijection $A \to B$. Why is it the case that $|\mathbb{N}| < |\mathbb{R}|$? In this way, there are at least two "sizes" for infinite sets.

Solution. There is a natural injection $f\colon \mathbb{N} \to \mathbb{R}$ given by $f(n) = n$, and the previous problem shows there is no bijection.

PROBLEM 3. Let A be a set and let $\mathcal{P}(A)$ be the set of all subsets of A. In this problem, we show that $|A| < |\mathcal{P}(A)|$. Thus, for instance, we see that

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \cdots .$$

(a) If $A = \{1, 2, 3\}$, find $\mathcal{P}(A)$.

(b) Describe an injection $A \to \mathcal{P}(A)$.

(c) We now show that there is no surjection $A \to \mathcal{P}(A)$. Let $f\colon A \to \mathcal{P}(A)$ be any function. Define

$$B = \{a \in A\colon a \notin f(a)\} .$$

We would like to show that B is not in the image of f, i.e., there is no $a \in A$ such that $f(a) = B$. For sake of contradiction, suppose there is an $a \in A$ such that $f(a) = B$. Then either $a \in B$ or $a \notin B$. Is $a \in B$? Is $a \notin B$?

Solution.

(a) We have

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\} .$$

(b) There are lots of them, but here is a natural one:

$$\begin{aligned} A &\mapsto \mathcal{P}(A) \\ a &\to \{a\} . \end{aligned}$$

(c) If $a \in B$, then since $B = f(a)$, we have $a \in f(a)$, which means $a \notin B$. So that cannot be. On the other hand, if $a \notin B$, then since $B = f(a)$, we have $a \notin B$, which means that $a \in B$. So that cannot be, either. It follows that there cannot be an a such that $f(a) = B$, and therefore, there is no surjection $A \to \mathcal{P}(A)$.

PROBLEM 1.

(a) If F is a field and x is a nonzero element of F. What is the meaning of $\frac{1}{x}$ (also denoted x^{-1})?

(b) What is $\frac{1}{3}$ in the field $\mathbb{Z}/7\mathbb{Z}$? (Denote the equivalence classes for $\mathbb{Z}/7\mathbb{Z}$ be $\{0,1,2,3,4,5,6\}$, for convenience.)

(c) Show that 2 does not have a multiplicative inverse in $\mathbb{Z}/8\mathbb{Z}$.

Solution.

(a) By $\frac{1}{x}$, we mean the multiplicative inverse of x, i.e., the element of F which when multiplied by x gives the multiplicative identity, 1.

(b) We have $\frac{1}{3} = 5$ in $\mathbb{Z}/7\mathbb{Z}$ since $3 \cdot 5 = 15 = 1 \bmod 7$.

(c) Here are the multiples of 2 modulo 8:

$$\begin{array}{c|cccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 2n & 0 & 2 & 4 & 6 & 0 & 2 & 4 & 6 \end{array}.$$

So there is no element of $n \in \mathbb{Z}/8\mathbb{Z}$ such that $2n = 1$ in $\mathbb{Z}/8\mathbb{Z}$.

PROBLEM 2. Let F be a field. In the lecture notes, we proved that for any $x \in F$, we have $x \cdot 0 = 0$. Using that proof as a model, prove that if $x, y, z \in F$, then

$$z + x = z + y \quad \Longrightarrow \quad x = y.$$

Your proof should proceed by using one axiom per step. (You will need A4, A2, and the definition of 0 (A3).) The above result is called the *cancellation law for addition* in a field.

Proof. Since F is a field, the element z has an additive inverse $-z$. Thus,

$$\begin{aligned} z + x = z + y \quad &\Rightarrow \quad -z + (z + x) = -z + (z + y) \\ &\Rightarrow \quad (-z + z) + x = (-z + z) + y \qquad \text{(associativity of } +\text{)} \\ &\Rightarrow \quad 0 + x = 0 + y \qquad \text{(definition of } -z\text{)} \\ &\Rightarrow \quad x = y \qquad \text{(definition of 0).} \end{aligned}$$

□

PROBLEM 3. Let F be a field, and let $x \in F$

(a) What is the meaning of $-x$?

(b) What is -3 in the field $\mathbb{Z}/7\mathbb{Z}$? (Again, denote the equivalence classes for $\mathbb{Z}/7\mathbb{Z}$ be $\{0,1,2,3,4,5,6\}$.)

(c) Prove that $-1 \cdot x = -x$. (You will need to focus on the definitions of -1 and $-x$. Since F is a field, it has a multiplicative identity 1, and that multiplicative identity must, like all element of F, have an additive inverse, -1. By definition, -1 is the element of F which when added to 1 yields the additive identity, 0. To test if a field element is $-x$, you add it to x and see if you get 0. You will also probably use the fact that $0 \cdot x = 0$, which we proved in the lecture notes.)

Solution.

(a) By $-x$, we mean the additive inverse of x, that is, the element of F which when added to x yields the additive identity, 0.

(b) We have $-3 = 4$ in $\mathbb{Z}/7\mathbb{Z}$ since $3 + 4 = 0$ in $\mathbb{Z}/7\mathbb{Z}$.

(c)

Proof. We have

$$\begin{aligned}
-1 \cdot x + x &= -1 \cdot x + 1 \cdot x && \text{(definition of 1)} \\
&= (-1 + 1) \cdot x && \text{(distributivity)} \\
&= 0 \cdot x && \text{(definition of } -1\text{)} \\
&= 0 && \text{(result from the lecture notes).}
\end{aligned}$$

Since adding $-1 \cdot x$ to x yields 0, it follows definition of the additive inverse that $-1 \cdot x = -x$. $\square$

Problem 1. Let F be an ordered field, and let $w, x, y, z \in F$. Use the order axioms to prove that if $w < x$ and $y < z$, then $w + y < x + z$. In other words, we can "add inequalities".

Proof. By additive translation, $w < x$ implies $w + y < x + y$. Similarly, $y < z$ implies $x + y < x + z$. Our result then follows by transitivity:

$$w + y < x + y \quad \text{and} \quad x + y < x + z \quad \implies \quad w + y < x + z.$$

□

Problem 2. Let F be an ordered field, and let $x \in F$ with $x > 0$. Since F is a field, x has a multiplicative inverse, $1/x$. Prove that $1/x > 0$. [Hint: break the possibilities for $1/x$ into cases using trichotomy, and rule out two of those cases.]

Proof. By trichotomy, exactly one of the following holds:

$$\frac{1}{x} = 0, \quad \frac{1}{x} < 0, \quad \text{or} \quad \frac{1}{x} > 0.$$

We will prove the result by ruling out the first two possibilities. First, using the definition of $1/x$ and the fact that $x \cdot 0 = 0$, as shown previously, note that

$$\frac{1}{x} = 0 \quad \Rightarrow \quad x \cdot \frac{1}{x} = x \cdot 0 \quad \Rightarrow \quad 1 = 0.$$

This cannot be, since $1 \neq 0$ in any field (as dictated explicitly in the definition of a field). Thus, $1/x \neq 0$.

Next, using multiplicative translation and the fact that $x > 0$,

$$\frac{1}{x} < 0 \quad \Rightarrow \quad x \cdot \frac{1}{x} < 0 \quad \Rightarrow \quad 1 < 0.$$

However, we have seen that in any field, $1 > 0$.

By process of elimination, we have $1/x > 0$. □

Problem 3. Can the field $\mathbb{Z}/5\mathbb{Z}$ be ordered? In other words, does there exist a relation on $\mathbb{Z}/5\mathbb{Z}$ satisfying the order axioms? [Hint: from the lecture notes, we know that for any nonzero element x of an ordered field, we have $x^2 > 0$. In particular, this means that $1 > 0$ since $1 = 1^2$. Start with $1 > 0$.]

Solution. For convenience, denote the elements of $\mathbb{Z}/5\mathbb{Z}$ by $0, 1, 2, 3, 4$, dropping the usual square brackets.

For the sake of contradiction, suppose that $\mathbb{Z}/5\mathbb{Z}$ could be ordered. We would then have $0 < 1$, and by repeated use of additive translation,

$$0 < 1 \Rightarrow 1 < 2 \Rightarrow 2 < 3 \Rightarrow 3 < 4 \Rightarrow 4 < 0.$$

By repeated application of transitivity, it would then follow that $0 < 0$, which violates trichotomy.

Recall the interval notion for subsets of the reals:

$$(a,b) := \{x \in \mathbb{R} : a < x < b\}, \qquad [a,b) := \{x \in \mathbb{R} : a \leq x < b\}, \qquad (a,b] := \{x \in \mathbb{R} : a < x \leq b\},$$
$$[a,b] := \{x \in \mathbb{R} : a \leq x \leq b\}, \qquad (-\infty,b) := \{x \in \mathbb{R} : x < b\}, \qquad (-\infty,b] := \{x \in \mathbb{R} : x \leq b\},$$
$$(a,\infty) := \{x \in \mathbb{R} : x > a\}, \qquad [a,\infty) := \{x \in \mathbb{R} : x \geq a\}, \qquad (\infty,\infty) := \mathbb{R}.$$

Recall the following definitions pertaining to a subset S of an ordered field F:

» $B \in F$ is an *upper bound* for S if $s \leq B$ for all $s \in S$,

» $b \in F$ is an *lower bound* for S if $b \leq s$ for all $s \in S$,

» S is *bounded* if it has both an upper bound and a lower bound.

» $B \in F$ is a *supremum* for S if it is a least upper bound. This means that B is an upper bound and if B' is any upper bound, then $B \leq B'$. If B exists, then we write $B = \sup(S)$ or $B = \operatorname{lub}(S)$.

» $b \in F$ is a *infimum* for S if it is a greatest lower bound. This means that b is a lower bound and if b' is any lower bound, then $b' \leq b$. If b exists, then we write $b = \inf(S)$ or $b = \operatorname{glb}(S)$.

» If S has a supremum B and $B \in S$, then we call B the *maximum* or *maximal element* of S and write $\max(S) = B$.

» If S has in infimum b and $b \in S$, then we call b the *minimum* of *minimal element* of S and write $\min(S) = b$.

Finally, recall that $\mathbb{R}$ satisfies the *completeness axiom*: every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum.

PROBLEM 1. Let $S = [0, 1) \subset \mathbb{R}$.

(a) Give three upper bounds and three lower bounds for S.

(b) Is S bounded? (Appeal to the definition of *bounded* here.)

(c) Does S have a supremum? If so, what is it? Same question for infimum.

(d) Does S have a maximum? a minimum?

Solution.

(a) For example, $1, 7$ and 10^6 are upper bounds and $0, -3$, and -23 are lower bounds.

(b) Yes, since S is bounded above and bounded below.

(c) The supremum of S is 1 and the infimum is 0.

(d) Since $\sup(S) = 1 \notin S$, the set S has no maximum. On the other hand, $\inf(S) = 0 \in S$. Thus, $\min(S) = 0$.

PROBLEM 2. These questions concern the ordered field of rational numbers $\mathbb{Q}$, not the field $\mathbb{R}$. Let $S = (0, \pi) \cap \mathbb{Q}$, a subset of $\mathbb{Q}$.

(a) Is S bounded?

(b) Does S have a supremum?

Solution.

(a) Yes. For instance, $4 \in \mathbb{Q}$ is an upper bound and $0 \in \mathbb{Q}$ is a lower bound.

(b) Since $\pi \notin \mathbb{Q}$ the set S has no supremum (in $\mathbb{Q}$).

PROBLEM 3. Here we're are considering subsets of $\mathbb{R}$. Fill in the following table, using "DNE" if the quantity does not exist:

	sup	max	inf	min
$[-1, 2)$				
$(-1, 2) \cup [3, 4]$				
$[3, \infty) \cup [3, 4]$				
$\mathbb{Z}_{\geq 0}$				
$\{-7, \sqrt{2}, 8, 23\}$				
$\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$				
$\cap_{n=1}^{\infty}(1 - 1/n, 1 + 1/n)$				
$\cup_{n=1}^{\infty}(1 - 1/n, 1 + 1/n)$				

.

Solution.

	sup	max	inf	min
$[-1,2)$	2	DNE	-1	-1
$(-1,2)\cup[3,4]$	4	4	-1	DNE
$[3,\infty)\cup[3,4]$	DNE	DNE	-1	DNE
$\mathbb{Z}_{\geq 0}$	DNE	DNE	0	0
$\{-7,\sqrt{2},8,23\}$	23	23	-7	-7
$\left\{\frac{n}{n+1} : n\in\mathbb{N}\right\}$	1	DNE	$\frac{1}{2}$	$\frac{1}{2}$
$\cap_{n=1}^{\infty}(1-1/n,1+1/n)$	1	1	1	1
$\cup_{n=1}^{\infty}(1-1/n,1+1/n)$	2	DNE	0	DNE

Note that $\cap_{n=1}^{\infty}(1-1/n,1+1/n)=\{1\}$, and $\cup_{n=1}^{\infty}(1-1/n,1+1/n)=(0,2)$.

PROBLEM 4. Mark each of the following statements as true or false. In each case, give a brief explanation if it is true or a specific counterexample if it is false. Throughout, S denotes a nonempty subset of $\mathbb{R}$.

(a) If S has an upper bound, then S has a least upper bound.

(b) If S is bounded, then S has a maximum and a minimum.

(c) If $S\subseteq\mathbb{Q}$ and S is bounded, then $\sup S\in\mathbb{Q}$.

(d) If $m=\inf S$ and $m'<m$, then m' is a lower bound of S.

Solution.

(a) True by the completeness axiom.

(b) False. A counterexample is $(0,1)$.

(c) False. A counterexample is given in an earlier problem: $(0,\pi)\cap\mathbb{Q}$.

(d) True. If $s\in S$, then it follows from the definition of the infimum that $m<s$. If $m'<m$, then by transitivity of $<$, we have $m'<s$, too.

Recall the following definitions pertaining to a subset S of an ordered field F:

» $B \in F$ is an *upper bound* for S if $s \leq B$ for all $s \in S$,

» $b \in F$ is an *lower bound* for S if $b \leq s$ for all $s \in S$,

» S is *bounded* if it has both an upper bound and a lower bound.

» $B \in F$ is a *supremum* for S if it is a least upper bound. This means that B is an upper bound and if B' is any upper bound, then $B \leq B'$. If B exists, then we write $B = \sup(S)$ or $B = \operatorname{lub}(S)$.

» $b \in F$ is a *infimum* for S if it is a greatest lower bound. This means that b is a lower bound and if b' is any lower bound, then $b' \leq b$. If b exists, then we write $b = \inf(S)$ or $b = \operatorname{glb}(S)$.

» If S has a supremum B and $B \in S$, then we call B the *maximum* or *maximal element* of S and write $\max(S) = B$.

» If S has in infimum b and $b \in S$, then we call b the *minimum* of *minimal element* of S and write $\min(S) = b$.

Recall that $\mathbb{R}$ satisfies the *completeness axiom*: every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum.

PROBLEM 1. Here were are considering subsets of $\mathbb{R}$. Fill in the following table, using "DNE" if the quantity does not exist:

	sup	max	inf	min
$\left\{\frac{1}{2n} : n \in \mathbb{N}_{>0}\right\}$				
$\left\{(-1)^n\left(1+\frac{1}{n}\right) : n \in \mathbb{N}_{>0}\right\}$				

.

Solution.

	sup	max	inf	min
$\left\{\frac{1}{2n} : n \in \mathbb{N}_{>0}\right\}$	$\frac{1}{2}$	$\frac{1}{2}$	0	DNE
$\left\{(-1)^n\left(1+\frac{1}{n}\right) : n \in \mathbb{N}_{>0}\right\}$	$\frac{3}{2}$	$\frac{3}{2}$	-2	-2

.

PROBLEM 2. Mark each of the following statements as true or false. In each case, give a brief explanation if it is true or a specific counterexample if it is false. Throughout, S denotes a nonempty subset of $\mathbb{R}$.

(a) If $B = \sup S$ and $B' < B$, then B' is an upper bound of S.

(b) If $B = \sup S$ and $B < B'$, then B' is an upper bound of S.

(c) $\emptyset$ is bounded.

(d) $\sup \emptyset$ and $\inf \emptyset$ do not exist.

Solution.

(a) False. A counter example is given by $S = (0, 1)$, $B = 1$ and $B' = 1/2$.

(b) True. Suppose $B < B'$. To see B' is an upper bound, let $s \in S$. By definition of the supremum, $s < B$. Then, by transitivity of $<$ it follows that $s < B'$.

(c) Yes. Every real number is both an upper bound and a lower bound for $\emptyset$. For instance, 3 is an upper bound since it is true that $3 > x$ for all $x \in \emptyset$. That's because there there exists no element x in $\emptyset$. Similar reasoning shows that 3 is also a lower bound.

(d) Since every real number is an upper bound for $\emptyset$, it follows that $\emptyset$ has no least upper bound, i.e., it has no supremum. A similar argument shows that $\emptyset$ does not have an infimum.

PROBLEM 3. Your answer to the last two parts of the previous problem shows that $\mathbb{R}$ has a subset that is bounded above but that has no supremum. Why doesn't that contradict the fact that $\mathbb{R}$ is complete.

Solution. The completeness axiom requires that every *nonempty* subset of $\mathbb{R}$ that is bounded above have a supremum.

Problem 4. Suppose that $\emptyset \neq X \subseteq S \subset \mathbb{R}$ and S has an supremum. Prove that

(a) $\sup X$ exists, and

(b) $\sup X \leq \sup S$.

(Hint for part (a): By completeness, you just need to show what about X? What could possibly be an upper bound for X? Hint for part (b): why do you just need to show that $\sup S$ is an upper bound for X?)

Proof.

(a) We first show that X is bounded above by $\sup(S)$. Let $x \in X$. Then, since $X \subseteq S$, we have $x \in S$, and hence $x \leq \sup(S)$. Thus, X is bounded above. Since $X \neq \emptyset$, it follows that from completeness of $\mathbb{R}$ that $\sup(X)$ exists.

(b) We have just shown that $\sup(S)$ is an upper bound for X. It follows from the definition of the supremum of X that $\sup(X) \leq \sup(S)$. (The idea is that $\sup(S)$ is *an* upper bound for X, and $\sup(X)$ is the *least* upper bound for X.)

$\square$

Problem 5. Let S be a subset of an ordered field F.

Recall the definition of the supremum: $B \in F$ is a *supremum* for S if it is a least upper bound. This means that B is an upper bound and if B' is any upper bound, then $B \leq B'$.

Use this definition to show that if u and v are both suprema of S, then $u = v$.

Proof. Suppose u and v are suprema of S. Then since u is *an* upper bound and v is a *least* upper bound, it follows that $v \leq u$. Similarly, since v is *an* upper bound, and u is a *least* upper bound, it follows that $u \leq v$.

Since $v \leq u$ and $u \leq v$, the trichotomy axiom for ordered fields implies that $u = v$. $\square$

PROBLEM 1. (Square roots of -1.) For $n \in \{2,3,4,5,6,10\}$, find all $x \in \mathbb{Z}/n\mathbb{Z}$ such that $x^2 = -1$.

Solution. In $\mathbb{Z}/2\mathbb{Z}$, we have $1^2 = 1 = -1$. In $\mathbb{Z}/5\mathbb{Z}$, we have $2^2 = 3^2 = 4 = -1$. In $\mathbb{Z}/10/\mathbb{Z}$, we have $3^2 = 7^2 = -1$. For the other values of n, there are no elements x such that $x^2 = -1$.

PROBLEM 2. Prove that $\mathbb{C}$ satisfies the additive associativity axiom. (Use the definition of $\mathbb{C}$, taking one step at a time, justifying each step. You will need to use the definition of addition for $\mathbb{C}$ and associativity of addition for $\mathbb{R}$.)

Proof. Let $(a,b),(a',b'),(a'',b'') \in \mathbb{C}$. Then

$$\begin{aligned}
\big((a,b)+(a',b')\big)+(a'',b'') &= (a+a',b+b')+(a'',b'') && \text{(def. of addition for } \mathbb{C})\\
&= ((a+a')+a'',(b+b')+b'') && \text{(def. of addition for } \mathbb{C})\\
&= (a+(a'+a''),b+(b'+b'')) && \text{(assoc. of addition for } \mathbb{R})\\
&= (a,b)+(a'+a'',b'+b'') && \text{(def. of addition for } \mathbb{C})\\
&= (a,b)+\big((a',b')+(a'',b'')\big) && \text{(def. of addition for } \mathbb{C})
\end{aligned}$$

□

PROBLEM 3. Consider the set $\mathbb{R}^2$ with addition and multiplication defined by

$$(a,b)+(c,d) = (a+c,b+d) \qquad \text{and} \qquad (a,b)\cdot(c,d) = (ac,bd),$$

respectively. Indicate which field axioms fail, giving a concrete counter-example in each case. What are the additive and multiplicative identities? (To save time, you may assume the fact that associativity of addition and multiplication hold.)

*Solution.*The existence of multiplicative inverses fails. First, note that $(0,0)$ is the additive identity and $(1,1)$ is the multiplicative identity. The element $(1,0)$ is nonzero, because $(1,0) \neq (0,0)$, yet $(1,0)$ has no multiplicative inverse: if $(a,b) \in \mathbb{R}^2$ satisfies

$$(1,1) = (1,0)\cdot(a,b) = (a,0),$$

then we must have $1 = 0$ in $\mathbb{R}$, which is absurd.

PROBLEM 4. Compute the following, expressing your result in the form $a+bi$ for $a,b \in \mathbb{R}$.

(a) $(3+2i)(-2+3i)+(1+4i)$

(b) $(2+3i)^{-1}$

(c) $\frac{1+4i}{2+i}$.

Solution.

(a)

$$(3+2i)(-2+3i)+(1+4i) = (-12+5i)+(1+4i) = -11+9i.$$

(b)

$$\frac{1}{2+3i} = \frac{1}{2+3i} \cdot \frac{2-3i}{2-3i} = \frac{2-3i}{2^2+3^2} = \frac{2}{13} - \frac{3}{13}i.$$

(c)

$$\frac{1+4i}{2+i} = \frac{1+4i}{2+i} \cdot \frac{2-i}{2-i} = \frac{6+7i}{2^2+1^2} = \frac{6}{5} + \frac{7}{5}i.$$

PROBLEM 1. Compute and write in standard form ($a + bi$ with $a, b \in \mathbb{R}$):

(a) $\overline{9 - 6i}$
(b) $|-3 + 2i|$
(c) $(-3 + 2i)^2$
(d) $(1 + i)/(1 - i)$
(e) $\operatorname{Im}((1 + i)/(1 - i))$.

Solution.

(a) $9 + 6i$
(b) $\sqrt{13}$
(c) $5 - 12i$
(d) i
(e) 1.

PROBLEM 2. Let $z = \cos(\theta) + \sin(\theta)i$ for some $\theta \in [0, 2\pi)$.

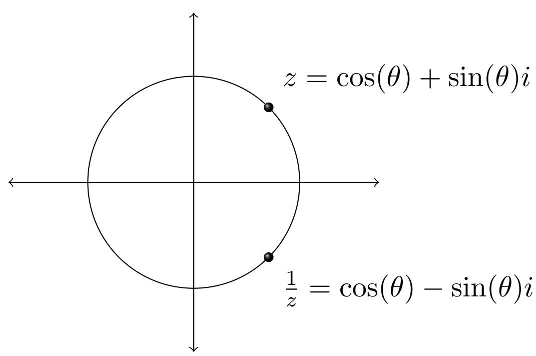
(a) Express $1/z$ in the form $a + bi$ with $a, b \in \mathbb{R}$.
(b) Plot z and $1/z$ for various values of θ. How are z and $1/z$ related geometrically?

Solution.

(a) We have

$$\begin{aligned}\frac{1}{\cos(\theta) + \sin(\theta)i} &= \frac{1}{\cos(\theta) + \sin(\theta)i}\frac{\cos(\theta) - \sin(\theta)i}{\cos(\theta) - \sin(\theta)i} \\ &= \frac{\cos(\theta) - \sin(\theta)i}{\cos^2(\theta) + \sin^2(\theta)} \\ &= \cos(\theta) - \sin(\theta)i.\end{aligned}$$

(b) The multiplicative inverse of $z = \cos(\theta) + \sin(\theta)i$ is obtained by reflecting z across the x-axis:



Problem 3. Let $z = (\sqrt{2}/2, \sqrt{2}/2)$. Compute and plot z^n in the plane for $n \geq 0$. (By definition $z^0 = 1$. Plot $1, z, z^2, z^3, \ldots,$ in turn. A pattern will eventually arise.)

Solution.

$$z^0 = 1 = (1, 0)$$

$$z^1 = (\sqrt{2}/2, \sqrt{2}/2)$$

$$z^2 = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2 = i = (0, 1)$$

$$z^3 = z \cdot z^2 = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) i = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$z^4 = z^2 \cdot z^2 = i^2 = -1 = (-1, 0)$$

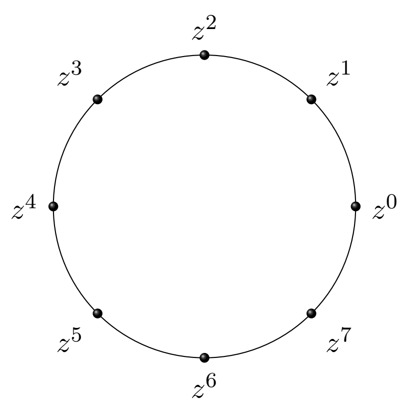
$$z^5 = z \cdot z^4 = -z = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

$$z^6 = z^2 \cdot z^4 = -z^2 = -i = (0, -1)$$

$$z^7 = z^3 \cdot z^4 = -z^3 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

$$z^8 = z^4 \cdot z^4 = -z^4 = -(-1) = 1 = (1, 0).$$

Here is a plot:



The point z is called an *eighth root of unity.* It's eighth power is 1 (and no lower power is 1). Note also that $z^2 = i$. So z is a square root of a square root of -1.

Problem 4. Let $z \in \mathbb{C}$. Prove that $|z| \geq |\operatorname{Im}(z)|$.

Proof. Say $z = a + bi$. We have

$$|z| = \sqrt{a^2 + b^2} \geq \sqrt{b^2} = |b| = |\operatorname{Im}(z)|.$$

$\square$

Let $n \geq 1$. A solution $z \in \mathbb{C}$ to the equation $z^n = 1$ is called an *n-th root of unity*. Our goal is to find all of them.

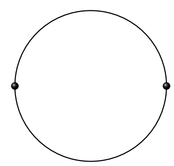
PROBLEM 1. For each $n \in \{2, 3, 4\}$, (i) find all $z \in \mathbb{C}$ such that $z^n = 1$ using algebra, (ii) find the polar form for each solution, and (iii) draw the solutions in the complex plane. (The case for each n should be on a separate page, finding the solutions to $z^n - 1 = 0$.)

(a) $n = 2$
(b) $n = 3$ (Hint: $z-1$ is a factor of z^3-1. So $z^3-1 = (z-1)(az^2+bz+c)$ for some $a, b, c \in \mathbb{C}$. Long division could help.)
(c) $n = 4$ (Hint: factor!)

Solution.

(a) $z^2 - 1 = (z-1)(z+1) = 0$ has solutions $z = \pm 1$. In polar form:

$$1 = \cos(0) + \sin(0)i, \quad -1 = \cos(\pi) + \sin(\pi)i$$

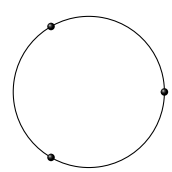


(b) $z^3 - 1 = (z-1)(z^2+z+1) = 0$ when $z = 1$ or when $z^2 + z + 1 = 0$. Use the quadratic equation to find the solutions it the latter:

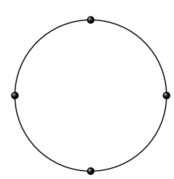
$$z = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\, i.$$

Polar forms

$$\begin{aligned} 1 &= \cos(0) + \sin(0)i \\ -\frac{1}{2} + \frac{\sqrt{3}}{2}\, i &= \cos(2\pi/3) + \sin(2\pi/3) \\ -\frac{1}{2} - \frac{\sqrt{3}}{2}\, i &= \cos(8\pi/6) + \sin(8\pi/6). \end{aligned}$$



(c) $z^4 - 1 = (z^2+1)(z^2-1) = 0$ if and only if $z^2 = -1$ or $z^2 = 1$. So the solutions are $\pm i$ and z:



Problem 2. If $z^n = 1$ for some $n \geq 1$, prove that z lies on the unit circle in the complex plane.

Proof. We have

$$z^n = 1 \quad \Rightarrow \quad |z^n| = |1| \quad \Rightarrow \quad |z|^n = 1 \quad \Rightarrow \quad |z| = 1.$$

□

Problem 3. Use the intuition you have developed so far to find the polar forms for all n-th roots of unity.

Solution. The n-th roots of unity are

$$\cos\left(\frac{2k\pi}{n}\right) + \sin\left(\frac{2k\pi}{n}\right) i$$

for $k = 0, 1, \ldots, n-1$.

Problem 4. (If there is extra time.) Let $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, and consider the mapping

$$\begin{aligned} f \colon \mathbb{C}^* &\to \mathbb{C} \\ z &\mapsto \frac{1}{\bar{z}}. \end{aligned}$$

(a) How does this mapping transform the modulus and the argument of each point in $\mathbb{C}^*$.

(b) Think of f as a geometric transformation. It takes the punctured plane $\mathbb{C}^*$, warps it somehow, and sends it back to the plane $\mathbb{C}^*$. Describe the process in geometric terms.

(Hint: writing z in polar form will help.)

Solution.

(a) Write $z \in \mathbb{C}^*$ in polar form as $z = |z|(\cos(\theta) + i\sin(\theta)$. Then

$$\begin{aligned} \frac{1}{\bar{z}} &= \frac{1}{|z|(\cos(\theta) - \sin(\theta))} \\ &= \frac{1}{|z|} \frac{1}{\cos(\theta) - i\sin(\theta)} \\ &= \frac{1}{|z|} \frac{1}{\cos(\theta) - i\sin(\theta)} \frac{\cos(\theta) + i\sin(\theta)}{\cos(\theta) + i\sin(\theta)} \\ &= \frac{1}{|z|}(\cos(\theta) + i\sin(\theta)). \end{aligned}$$

Thus,

$$|f(z)| = \frac{1}{|z|} \quad \text{and} \quad \arg(f(z)) = \arg(z).$$

For instance, the image of a circle of radius r centered at the origin is again a circle, but of radius $1/r$. Very small circles about the origin are sent to very large circles, and vice versa. In this way, the plane is turned inside out.

Problem 5. (If there is extra time.) The graph of a function $f\colon A \to B$ is the set $\{(x, f(x)) \in A \times B\}$. Thus, if $A = B = \mathbb{R}$, we can draw the graph in $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$. What if f is a complex function, i.e., if $A = B = \mathbb{C}$? Then the graph sits in $\mathbb{C} \times \mathbb{C} = \mathbb{R}^2 \times \mathbb{R}^2 \simeq \mathbb{R}^4 = \{(a, b, c, d) : a, b, c, d \in \mathbb{R}\}$. Since the graph of a function sits in four-dimensional space, we need a different strategy to picture these functions. Here is one way. Suppose $f(z) = (a, b) \in \mathbb{C} = \mathbb{R}^2$. We encode $a \in \mathbb{R}$ as color, and $b \in \mathbb{R}$ as brightness. (To do that, we need to choose reasonable functions $a \mapsto$ color and $b \mapsto$ brightness.) To picture the graph of f, we go to each point $z \in \mathbb{C} = \mathbb{R}^2$, and color that point with its corresponding color and brightness.

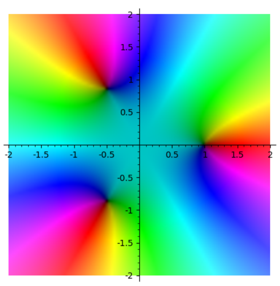
Go to https://sagecell.sagemath.org/ and type in the following to see the graph of $f(z) = z^3 - 1$:

```
p = complex_plot(lambda z: z^3-1, (-2, 2), (-2, 2))
p.show(aspect_ratio=1)
```

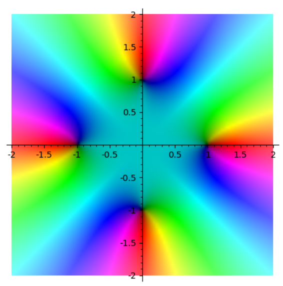
(a) Try graphing the functions $z^n = 1$ for various values of n. How do you connect these pictures to the roots of unity?

(b) Try graphing some other functions, e.g, `z^2+2*z+1` (note the $*$ for multiplication). A polynomial of degree 3? What about trig functions? The mysterious Riemann zeta function `zeta(z)`? For the zeta function, you might want to zoom out by replacing `(-2,2), (-2,2)` by `(-20,20), (-20,20)`.

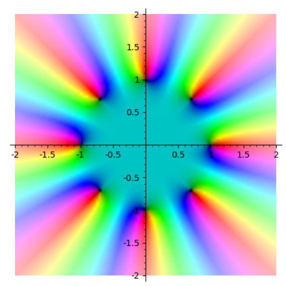
Solution.



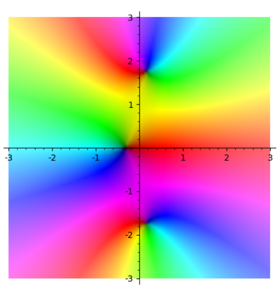
$z^3 - 1$



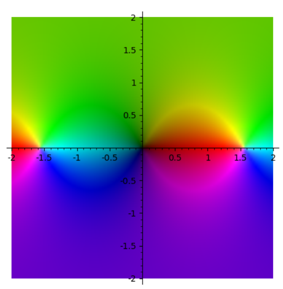
$z^4 - 1$



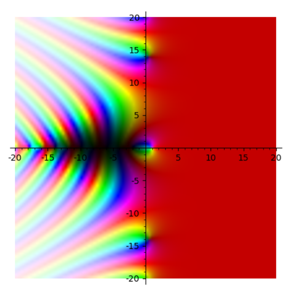
$z^8 - 1$



$z^3 + 2z + z + 1$



$\tan(z)$



$\zeta(z)$

In these problems, let $F = \mathbb{R}$ or $\mathbb{C}$. Your proofs should work simultaneously for both. To find your proofs, it might be easier to draw pictures in the complex plane.

Definition. A subset $U \subseteq F$ is *open* if it contains an open ball about each of its points. This means that for all $u \in U$, there exists $\varepsilon > 0$ such that

$$B(u; \varepsilon) \subseteq U,$$

i.e., if $w \in F$ and $|w - u| < \varepsilon$, then $w \in U$.

Proof template. Let U be the subset of F defined by blah, blah, blah. Then U is open.

Steps in direct proof: (1) Let $u \in U$. (2) What should ε be? (3) Argue that $B(u, \varepsilon) \subseteq U$, i.e., that $|w - u| < \varepsilon$ implies $w \in U$. $\square$

PROBLEM 1. Let $z \in \mathbb{C}$. Prove that $\mathbb{C} \setminus \{z\}$ is open. (Hints: Given a point w in the set, what should ε be? Why is the resulting open ball of radius ε about w contained in the original set? Write this down using complete sentences.)

Proof. Given $w \in \mathbb{C} \setminus \{z\}$, let $\varepsilon := |w - z|$. We claim that $B(w; \varepsilon) \subseteq \mathbb{C} \setminus \{z\}$. It suffices to show that $z \notin B(w; \varepsilon)$. To see this, note that

$$|z - w| =: \varepsilon \not< \varepsilon. \qquad \square$$

PROBLEM 2. In any topology, the intersection of a finite number of open sets is open. Let $U_1, \dots, U_k$ be open subsets of $\mathbb{R}$ or $\mathbb{C}$. Prove that $\cap_{i=1}^k U_i$ is open. (Hints: Given a point w in the set, what should ε be? Why is the resulting open ball of radius ε about w contained in the original set? Write this down using complete sentences.)

Proof. Let $u \in \cap_{i=1}^k U_i$. Then $u \in U_i$ for $i = 1, \dots, k$. For each i, since U_i is open, there exists $\varepsilon_i > 0$ such that $B(u; \varepsilon_i) \subseteq U_i$. Define $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_k\}$. Then $\varepsilon > 0$. We claim $B(u; \varepsilon) \subseteq \cap_{i=1}^k U_i$. To see this, let $w \in B(u; \varepsilon)$. For each i, we have

$$|w - u| < \varepsilon \leq \varepsilon_i.$$

Hence, $w \in B(u; \varepsilon_i) \subseteq U_i$ for each i, and so $w \in U_i$. Since $w \in U_i$ for all i, it follows that $w \in \cap_{i=1}^k U_i$. $\square$

PROBLEM 3. Let I be any index set, and for each $i \in I$, let U_i be an open subset of F. Is $\cap_{i \in I} U_i$ necessarily open? Give a proof or a (concrete) counterexample.

SOLUTION: We have just shown that finite intersections of open sets are open. However, an infinite intersection of open sets is not necessarily open. For example, let $\varepsilon_i = 1/i$, and let $U_i = B(0, \varepsilon_i) \subset \mathbb{C}$ for $i = 1, 2, 3, \dots$ Then $\cap_{i \geq 1} U_i = \{0\} \in \mathbb{C}$. The set $\{0\}$ is not open since every open set containing $\{0\}$ contains nonzero points, too. $\square$

PROBLEM 1. Let $a_n = (-1)^n$, and let $a = 0$. Is the following statement true or false? Provide a proof or explicit counterexample.

For all $N \in \mathbb{R}$ there is an $\varepsilon > 0$, such that if $n > N$, then $|a - a_n| < \varepsilon$.

What is the relevance of the above statement to the question of the convergence or divergence of $\{a_n\}$?

SOLUTION: The statement is true. For instance, take $\varepsilon = 10$. Then

$$|a - a_n| = |0 - (-1)^n| = 1 < 10 = \varepsilon$$

for all n. So no matter the choice of N, we have $n > N$ implies that $|a - a_n| < 10$.

The statement, although vaguely similar to the definition of the limit, is irrelevant to the question of convergence.

PROBLEM 2. Let $a_n = 1/n$ for $n \geq 1$, and let $a = 0$. Is the following statement true or false? Provide a proof or explicit counterexample.

For all $\varepsilon > 0$ and $N \in \mathbb{R}$, if $n > N$, then $|a - a_n| < \varepsilon$.

What is the relevance of the above statement to the question of the convergence or divergence of $\{a_n\}$?

SOLUTION: The statement is false. For instance, let $\varepsilon = 1/2$, $N = 0$, and $n = 1$. Then $n > N$, but

$$|a - a_n| = |a - a_1] = |0 - 1/1| = 1 \not< \frac{1}{2} = \varepsilon,$$

i.e., $|a - a_1| \not< \varepsilon$.

Again, although the statement is vaguely similar to the definition of the limit, is irrelevant to the question of convergence.

PROBLEM 3. Find the limit of $\lim\limits_{n\to\infty} \dfrac{3n^3 + 2n}{6n^3 + 4n + 7}$ and provide an ε-N proof.

SOLUTION: Claim: $\lim\limits_{n\to\infty} \dfrac{3n^3 + 2n}{6n^3 + 4n + 7} = \dfrac{1}{2}$.

Proof. Given $\varepsilon > 0$, let $N = \max\left\{1, \dfrac{7}{12\varepsilon}\right\}$ and suppose $n > N$. Then

$$\begin{aligned}\left|\frac{1}{2} - \frac{3n^3 + 2n}{6n^3 + 4n + 7}\right| &= \frac{(6n^3 + 4n + 7) - 2(3n^3 + 2n)}{2\,(6n^3 + 4n + 7)} \\ &= \frac{7}{12n^3 + 8n + 14}\end{aligned}$$

$$
\begin{aligned}
&< \frac{7}{12n^3} \\
&\le \frac{7}{12n} \\
&< \frac{7}{12N} \\
&= \varepsilon.
\end{aligned}
$$

$\square$

PROBLEM 4. (Challenge, if there is extra time.) Prove that $\lim_{n\to\infty} \frac{1}{n} \neq 1$. (Hint: you need to find an explicit $\varepsilon > 0$ that can't be beat by any $N \in \mathbb{R}$.)

Proof. We have $a = 1$ and $a_n = 1/n$. So

$$|a - a_n| = \left|1 - \frac{1}{n}\right| = \frac{n-1}{n}.$$

The question is whether, given arbitrary $\varepsilon > 0$, can we find N such that $n > N$ implies $|a - a_n| < \varepsilon$? Since $|a - a_n|$ is getting close to 1 as n gets large, the answer is no—not for arbitrary ε. Let $\varepsilon = 1/2$, for instance. Then

$$|a - a_n| \ge \varepsilon \quad \iff \quad \frac{n-1}{n} \ge \frac{1}{2} \quad \iff \quad 2(n-1) \ge n \quad \iff \quad n \ge 2.$$

So no matter what the value of $N \in \mathbb{R}$, there will be an $n > N$ such that $|a - a_n| \not< 1/2$. $\square$

PROBLEM 1. Give an ε-N proof that

$$\lim_{n\to\infty} \frac{\cos(n) + \sqrt{2}i\sin(n)}{n} = 0.$$

(Hint: the triangle inequality is your friend.)

Proof. Given $\varepsilon > 0$, let $N = 3/\varepsilon$. If $n > N$, it follows that

$$\begin{aligned}
\left|0 - \frac{\cos(n) + \sqrt{2}i\sin(n)}{n}\right| &= \left|\frac{\cos(n) + \sqrt{2}i\sin(n)}{n}\right| \\
&= \frac{|\cos(n) + \sqrt{2}i\sin(n)|}{n} \\
&\leq \frac{|\cos(n)| + |\sqrt{2}i\sin(n)|}{n} \\
&= \frac{|\cos(n)| + \sqrt{2}|\sin(n)|}{n} \\
&\leq \frac{1+\sqrt{2}}{n} \\
&\leq \frac{3}{n} \\
&< \frac{3}{N} \\
&= \varepsilon.
\end{aligned}$$

□

PROBLEM 2. Give an ε-N proof that

$$\lim_{n\to\infty} \frac{n}{4n^3 + 2n^2 + 5n + 1} = 0.$$

Proof. Given $\varepsilon > 0$, let $N = \sqrt[3]{\varepsilon}$. If $n > N$, then

$$\left|0 - \frac{1}{4n^3 + 2n^2 + 5n + 1}\right| = \frac{1}{4n^3 + 2n^2 + 5n + 1} < \frac{1}{4n^3} < \frac{1}{n^3} < \frac{1}{N^3} = \varepsilon.$$

□

PROBLEM 3. Give an ε-N proof that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0.$$

Proof. Given $\varepsilon > 0$, let $N = 1/\varepsilon^2$. If $n > N$, then

$$\left|0 - \frac{1}{\sqrt{n+1}+\sqrt{n}}\right| = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}} < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} = \varepsilon.$$

□

PROBLEM 4. Does the sequence $\left\{\sqrt{n+1} - \sqrt{n}\right\}$ converge? Proof?

Solution. Claim: $\lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n}) = 0$.

Proof. Given $\varepsilon > 0$, let $N = 1/\varepsilon^2$. Then if $n > N$, we have

$$\begin{aligned}
|0 - (\sqrt{n+1} - \sqrt{n})| &= |\sqrt{n+1} - \sqrt{n}| \\
&= \left|\left(\frac{\sqrt{n+1} - \sqrt{n}}{1}\right) \cdot \left(\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}\right)\right| \\
&= \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} \\
&= \frac{1}{\sqrt{n+1} + \sqrt{n}} \\
&< \frac{1}{2\sqrt{n}} \\
&< \frac{1}{\sqrt{n}} \\
&< \frac{1}{\sqrt{N}} \\
&= \varepsilon.
\end{aligned}$$

□

PROBLEM 5. (Challenge, if you have extra time.)
Does $\{\frac{n!}{n^n}\}$ converge? (Hint: write $n!/n^n$ as a product of n distinct factors, and try to bound it above by a nice function of n.)

SOLUTION: Claim $\displaystyle\lim_{n\to\infty} \frac{n!}{n^n} = 0.$

Proof. Write $n!/n^n$ as the product of n distinct terms:

$$\frac{n!}{n^n} = \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{2}{n} \cdot \frac{1}{n}.$$

Notice that $k/n \leq 1$ for $k = 2, 3, \ldots, n$, so each of the first $n-1$ terms in the above product is bounded above by 1. It follows that

$$0 \leq \frac{n!}{n^n} \leq \frac{1}{n}$$

for $n \geq 1$.

Given $\varepsilon > 0$, let $N = 1/\varepsilon$. If $n > N$, then using what we have just learned,

$$\left| 0 - \frac{n!}{n^n} \right| = \frac{n!}{n^n} \leq \frac{1}{n} < \frac{1}{N} = \varepsilon.$$

$\square$

Dynamical Systems

self-mapping of a set S: a function $f\colon S \to S$.

n**-th iterate** of s under f:

$$f^n(s) := \begin{cases} s & \text{if } n = 0 \\ f(f^{n-1}(s)) & \text{if } n > 0. \end{cases}$$

orbit of s under f: $\mathrm{Orb}_f(s) := \{s, f(s), f^2(s), f^3(s), \ldots\}$ = the iterates of s under f.

fixed points of f: $\mathrm{Fix(f)} := \{s \in S : f(s) = s\}$.

In the following problems, let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 - 1$. Our goal is to understand the orbits of f.

Problem 1. What is the orbit of 0 under f? What is the orbit of -1?

Solution: We have $f(0) = -1$ and $f(f(0)) = f(-1) = 0$. Therefore, the orbit of 0 is $\mathrm{Orb}_f(0) = \{-1, 0\}$, and the orbit of -1 is $\mathrm{Orb}_f(-1) = \{-1, 0\}$.

Problem 2. What are the first four iterates of $\frac{1}{2}$, i.e., $f^0(1/2), f^1(1/2), f^2(1/2), f^3(1/2)$? (You do not need to evaluate.)

Solution: We have

$$\begin{aligned} f^0(1/2) &= 1/2 \\ f^1(1/2) &= (1/2)^2 - 1 = -3/4 \\ f^2(1/2) &= f(-3/4) = (-3/4)^2 - 1 = -7/16 \\ f^3(1/2) &= f(-7/16) = (-7/16)^2 - 1 = -207/256. \end{aligned}$$

Problem 3. Label the 12 dots in Figure 1 using the notation $f^i(1/2)$.

Solution: See Figure 1.

Problem 4. What are the fixed points of f? How can you picture these in Figure 1?

SOLUTION: We have $x = f(x) = x^2 - 1$, or $x^2 - x - 1 = 0$. The two solutions to this equation are

$$\frac{1 \pm \sqrt{5}}{2}.$$

We can visualize these values as the x-coordinates of the point of intersection of the line $y = x$ with the graph of f in Figure 1.

PROBLEM 5. Draw a picture as in Figure 1 with an initial point just to the left of the positive fixed point. See Figure 2.

SOLUTION: See Figure 2.

PROBLEM 6. Use induction to prove that if $x \in [-1, 0]$, then $f^n(x) \in [-1, 0]$ for all $n \geq 0$. (You may use standard facts about real numbers.)

SOLUTION: Let $x \in [-1, 0]$. The base case holds since $f^0(x) = x \in [-1, 0]$. Suppose that $a := f^n(x) \in [-1, 0]$ for some $n \geq 0$. Then

$$f^{n+1}(x) = f(f^n(x)) = f(a) = a^2 - 1.$$

Since $a \in [-1, 0]$, it follows that $a^2 \in [0, 1]$. (Details: We have $0 \leq -a < 1$, which implies that $0 \leq (-a)^2 \leq 1$.) Therefore, $f(a) = a^2 - 1 \in [-1, 0]$. The result follows by induction.

PROBLEM 7. Show that if $x \in [-1, 1]$, then $f(x) \in [-1, 0]$ for all $n \geq 1$.

SOLUTION: If $x \in [-1, 1]$, then $x^2 \in [0, 1]$, and hence, $f(x) = x^2 - 1 \in [-1, 0]$. The result then follows from the previous problem.

Facts. Let α denote the positive fixed point of f. Then:

- If $x \in (\alpha, \infty)$, then the iterates of x increase without bound. (So the orbit of x is unbounded.)
- If $x \in (-\infty, -\alpha)$, then $f(x) \in (\alpha, \infty)$.
- If $x \in (1, \alpha)$, then $f(^n)(x)$ decreases until an iterate is in $[0, 1]$.
- If $x \in (-\alpha, -1)$, then $f(x) \in (0, \alpha)$.

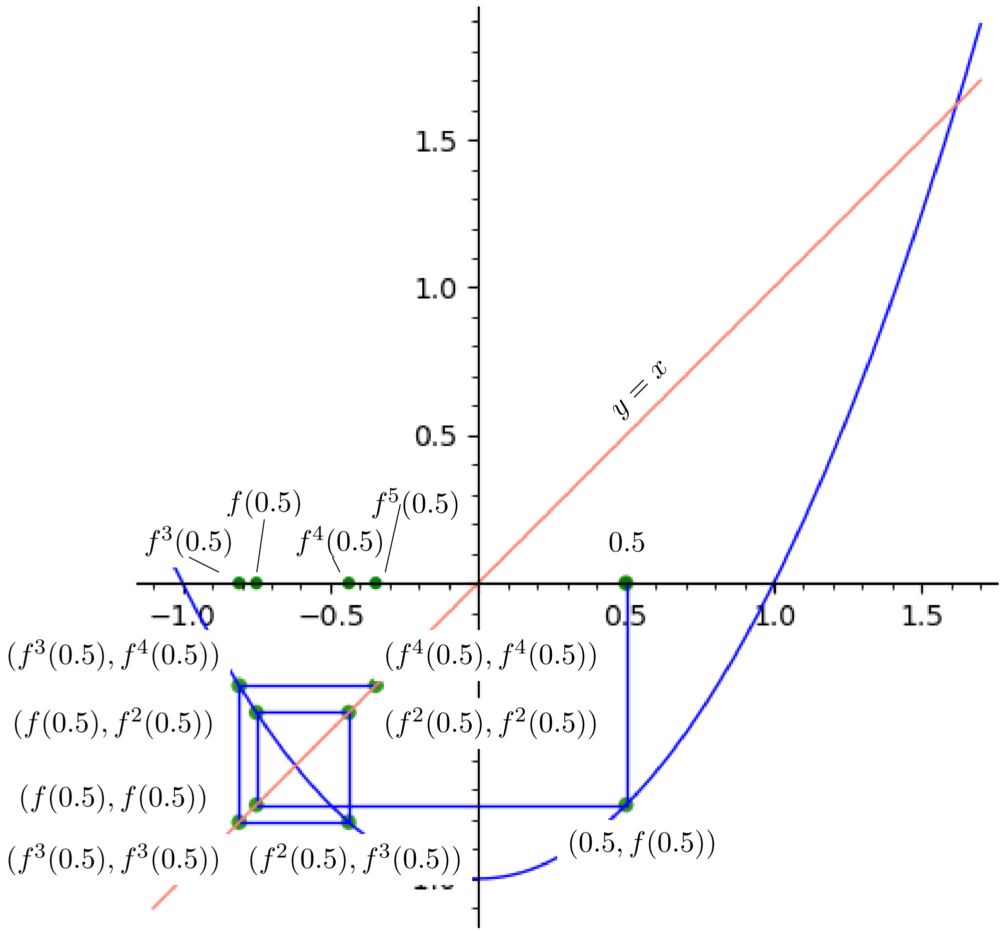


FIGURE 1. Visualizing the dynamical system determined by $f(x) = x^2 - 1$.

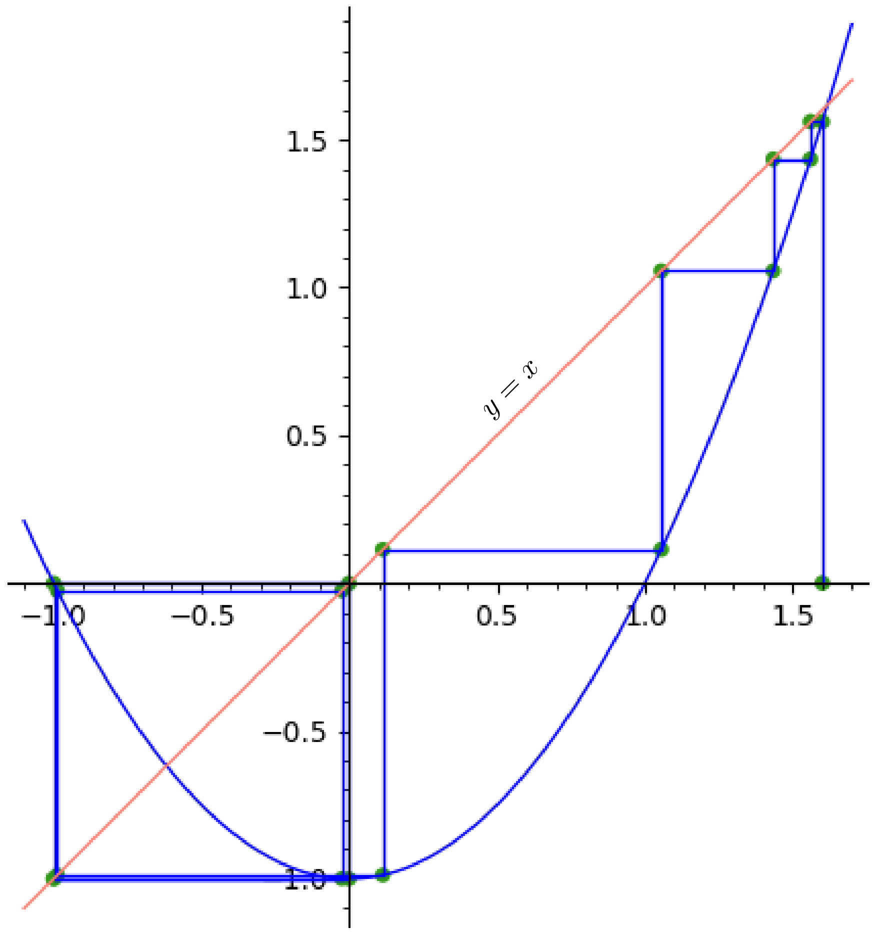


FIGURE 2. Draw a picture as in Figure 1 with an initial point just to the left of the positive fixed point.

Dynamical Systems

Let $S = \mathbb{R}$ or $\mathbb{C}$, and let $f: S \to S$. The *filled Julia set* for f is

$$K(f) = \{z \in S : \mathrm{Orb}_f(z) \text{ is bounded}\}.$$

Thus, $K(f)$ is the set of points $z \in S$ whose iterates are bounded: there exists a real number r such that $|f^n(z)| \leq r$ for all $n \geq 0$. The *Julia set*, denoted $J(f)$, is the boundary[1] of $K(f)$.

Example. Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 - 1$. We saw last time that $K(f) = [-\alpha, \alpha]$ where $\alpha = \frac{1+\sqrt{5}}{2}$. Thus, $J(f)$ consists of the two endpoints: $J(f) = \{-\alpha, \alpha\}$.

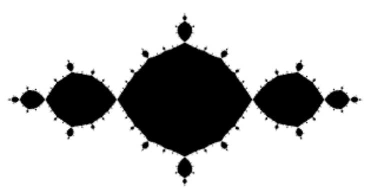
PROBLEM 1. What are the filled Julia set and the Julia set for the function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$?

SOLUTION: We have $K(f) = [-1, 1]$, and $J(f) = \{-1, 1\}$.

PROBLEM 2. What are the filled Julia set and the Julia set for the function $f: \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^2$?

SOLUTION: We have $K(f)$ is the closed disc of radius one centered at the origin, and $J(f)$ is the circle of radius one centered at the origin.

PROBLEM 3. Consider the filled Julia set $K(f)$ for the function $f: \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^2 - 1$. The set $K(f) \subseteq \mathbb{C}$ is pictured below:



(a) What is the horizontal line segment running through the middle (along the real axis)?

(b) Is $i \in K(f)$? What about $i/2$? (Hint: use that fact that you know something about the filled Julia set for f restricted to the real numbers. Note: $1.61 \leq (1+\sqrt{5})/2 \leq 1.62$.)

SOLUTION:

(a) $[-\alpha, \alpha]$ with α as defined above.

[1] If X is a subset of a topological space, the *closure* of X, denoted $\overline{X}$, is the smallest closed set containing K. It is the intersection of all closed set containing X. The *boundary* of X is the intersection of the closure of X and the closure of the complement of X. Example: the closure of an open ball in $\mathbb{C}$ is a circle.

(b) We have $f(i) = i^2 - 1 = -2 < -\alpha$. From this iterate onward, we may as well be considering the real version of f with filled in Julia set $[-\alpha, \alpha]$. Since $2 \notin [-\alpha, \alpha]$, its iterates are not bounded. Hence, $i \notin K(f)$. On the other hand, $f(i/2) = (i/2)^2 - 1 = -5/4 = -1.25 \in [-\alpha, \alpha]$. Hence, its iterates are bounded. Therefore, $i/2 \in K(f)$.

Problem 4. Let $c \in \mathbb{C}$ and consider the function $f_c\colon \mathbb{C} \to \mathbb{C}$ defined by $f_c(z) = z^2 + c$. (For instance, $f_{-1}(z) = z^2 - 1$.) Show that $K(f_c)$ is symmetric about the origin by showing that $z \in K(f) \Rightarrow -z \in K(f)$.

Solution: Since $f_c(-z) = (-z)^2 + c = z^2 + c = f(z)$, the iterates of z are bounded if and only if the iterates of $-z$ are bounded.

Problem 5. Go to https://www.marksmath.org/visualization/julia_sets/. There are two copies of $\mathbb{C}$ pictured on that page. Clicking a point on the left side selects a point $c \in \mathbb{C}$, and the number c is displaying in a box underneath. You can choose c without clicking by entering it in this box. The right side then shows the Julia set for $f_c(z) = z^2 + c$.

(a) Enter the point $c = 0$ to see the Julia set for $f_{-1}(z) = z^2$. (You will see the point displayed in the set on the left.)

(b) Enter the point $c = -1$ to see the Julia set for $f_{-1}(z) = z^2 - 1$.

(c) What happens as you click points along the real axis going from 0 to -1?

(d) Hit "Clear" to erase the Julia sets drawn so far. The shape pictured in the right is the Mandelbrot set, M. It is the set of points $c \in \mathbb{C}$ such that the iterates of 0 under the mapping $f_c(z) = z^2 + c$ are bounded, i.e., $0, c, c^2 + c, (c^2 + c)^2 + c, \ldots$ is bounded. What distinguishes Julia sets for $c \in M$ and $c \notin M$?

Problem 6. Show that $K(f_c)$ is symmetric about the real axis.

Problem 7. Prove that for all $c \in \mathbb{C}$, we have $K(f_c) \neq \emptyset$.

PROBLEM 1. Find the limit of the sequence $\left\{\frac{3n^2-5}{n^2-3n+2}\right\}$ using our limit theorems (i.e., without using an ε-N argument). Justify each step.

Solution. Using our limit theorems,

$$\begin{aligned}
\lim_{n\to\infty} \frac{3n^2-5}{n^2-3n+2} &= \lim_{n\to\infty} \frac{\frac{1}{n^2}(3n^2-5)}{\frac{1}{n^2}(n^2-3n+2)} \\
&= \lim_{n\to\infty} \frac{3-\frac{5}{n^2}}{1-\frac{3}{n}+\frac{2}{n^2}} \\
&= \frac{\lim_{n\to\infty} 3 + (\lim_{n\to\infty}(-5))\left(\lim_{n\to\infty}\frac{1}{n}\right)^2}{\lim_{n\to\infty} 1 + (\lim_{n\to\infty}(-3))\left(\lim_{n\to\infty}\frac{1}{n}\right) + (\lim_{n\to\infty} 2)\left(\lim_{n\to\infty}\frac{1}{n}\right)^2} \\
&= \frac{3-5\cdot 0^2}{1-3\cdot 0+2\cdot 0^2} \\
&= 3.
\end{aligned}$$

PROBLEM 2. We have shown that $\lim_{n\to\infty}\frac{\sin(n)}{n} = 0$. Use this result along with our limit theorems to find the limit of the sequence $\left\{\frac{\sin(n)}{n^2-n+1}\right\}$ justifying each step.

Solution. Using our limit theorems,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sin(n)}{n^2-n+1} &= \frac{\lim_{n\to\infty}\frac{1}{n^2}\sin(n)}{\lim_{n\to\infty}\frac{1}{n^2}(n^2-n+1)} \\
&= \frac{\lim_{n\to\infty}\frac{\sin(n)}{n^2}}{\lim_{n\to\infty}\left(1-\frac{1}{n}+\frac{1}{n^2}\right)} \\
&= \frac{\left(\lim_{n\to\infty}\left(\frac{\sin(n)}{n}\right)\right)\left(\lim_{n\to\infty}\frac{1}{n}\right)}{(\lim_{n\to\infty} 1) + (\lim_{n\to\infty}(-1))\left(\lim_{n\to\infty}\frac{1}{n}\right) + \left(\lim_{n\to\infty}\frac{1}{n}\right)^2} \\
&= \frac{0\cdot 0}{1-1\cdot 0+0^2} \\
&= 0.
\end{aligned}$$

PROBLEM 3. State whether each of the following statements is true or false (with proof or concrete counterexample):

(a) If $\{a_n\}$ and $\{b_n\}$ both diverge, then $\{a_n+b_n\}$ diverges.

(b) If $\{a_n\}$ converges and $\{b_n\}$ diverges, then $\{a_n+b_n\}$ diverges.

Solution.

(a) False. Consider $a_n = n$ and $b_n = -n$.

(b) True. Suppose $\{a_n\}$ converges. Then by the limit theorems, if $\{a_n + b_n\}$ converges, it follows that
$$\lim_{n\to\infty}(a_n + b_n) - \lim_{n\to\infty} a_n$$
exists and equals $\lim_{n\to\infty} b_n$. So $\{b_n\}$ would have to converge.

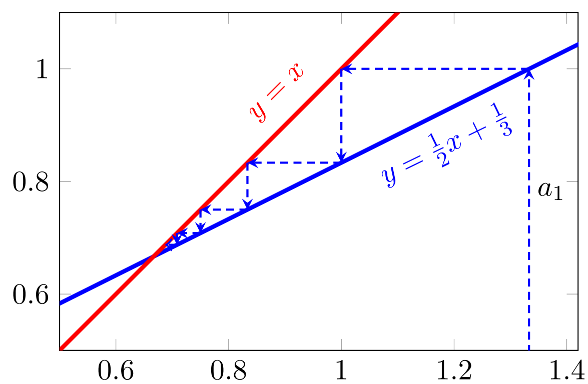
PROBLEM 4. Let $k \in \mathbb{N}_{>0}$. Find, with proof, the limit of the sequence $\left\{\left(\frac{n+1}{n}\right)^k\right\}$.

Solution. We find
$$\begin{aligned}
\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^k &= \left(\lim_{n\to\infty}\frac{n+1}{n}\right)^k \\
&= \left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\right)^k \\
&= \left(\lim_{n\to\infty} 1 + \lim_{n\to\infty}\frac{1}{n}\right)^k \\
&= (1+0)^k \\
&= 1^k \\
&= 1.
\end{aligned}$$

PROBLEM 5. Suppose that $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$. Review the proof that
$$\lim_{n\to\infty}(s_n + t_n) = s + t.$$

Proof. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} s_n = s$, there exists $N_s \in \mathbb{R}$ such that $n > N_s$ implies $|s - s_n| < \varepsilon/2$. Similarly, there exists $N_t \in \mathbb{R}$ such that $n > N_t$ implies $|t - t_n| < \varepsilon/2$. Let $N := \max\{N_s, N_t\}$. Then $n > N$ implies both $|s - s_n| < \varepsilon/2$ and $|t - t_n| < \varepsilon/2$, simultaneously. Using the triangle inequality, it follows that if $n > N$,
$$|(s+t) - (s_n + t_n)| = |(s - s_n) + (t - t_n)| \le |s - s_n| + |t - t_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
□

PROBLEM 1.

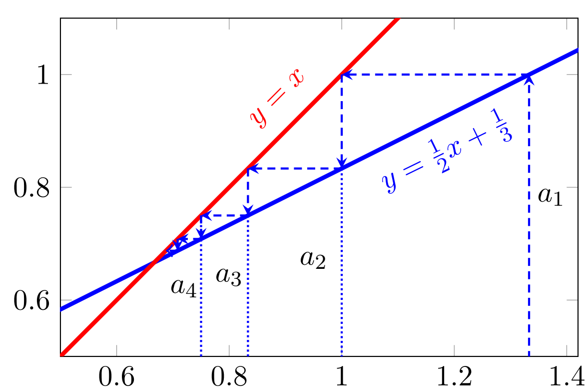


(a) Following the densely dashed blue line, how can you see the sequence $a_1 = 4/3$ and $a_{n+1} = \frac{1}{2}a_n + \frac{1}{3}$ for $n \geq 1$ as a sequence of heights in the above diagram?

(b) Using the diagram, can you say what the behavior of the sequence would be if we started with a different value for a_1 (but used the same recursion formula for the rest of the sequence)?

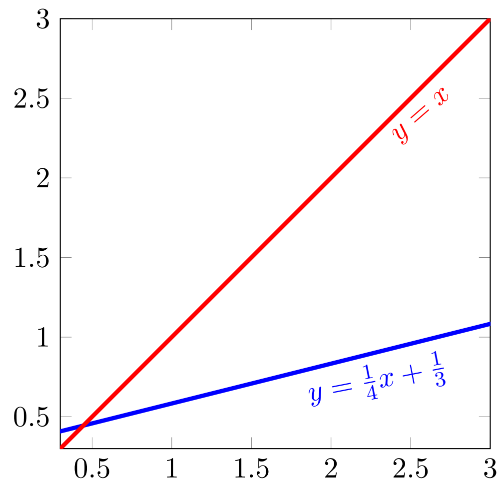
Solution.

(a)



(b) With any initial point, the sequence will converge to 2/3.

PROBLEM 2. Here is a diagram for the sequence $a_1 \in \mathbb{R}$ and $a_{n+1} = \frac{1}{4}a_n + \frac{1}{3}$.

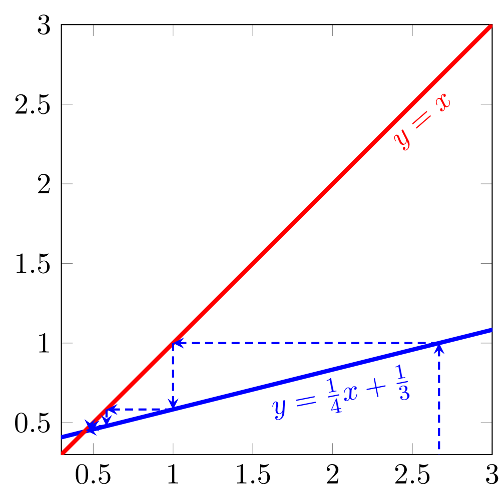


(a) Choose a value for a_1, and draw a diagram illustrating the convergence of $\{a_n\}$.

(b) Comparing this sequence with the previous one, how does the slope of the line defining the recurrence affect the rate of convergence.

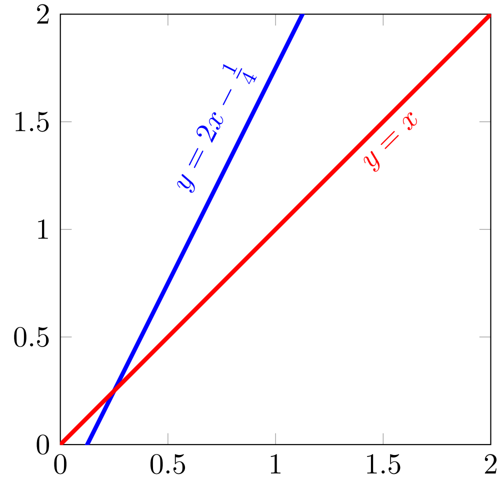
Solution.

(a)



(b) This larger slope causes faster convergence.

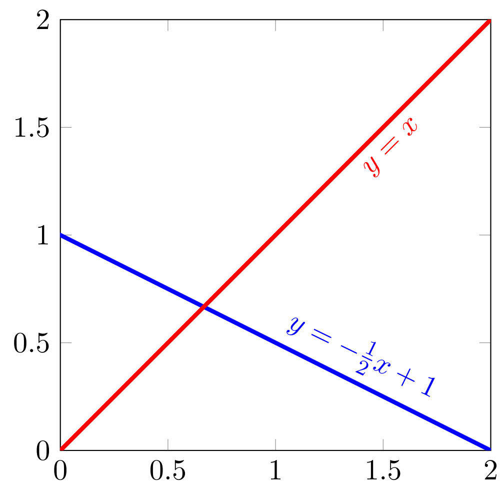
Problem 3. Here is a diagram for the sequence $a_1 \in \mathbb{R}$ and $a_{n+1} = 2a_n - \frac{1}{4}$.



Characterize the convergence behavior of the sequence $\{a_n\}$ for each choice of initial value a_1. (Are there any special choices for a_1?)

Solution. The sequence diverges for every initial value except $a_1 = 1/4$. When $a_1 = 1/4$, we get the constant sequence at $1/4$.

PROBLEM 4. Here is a diagram for the sequence $a_1 \in \mathbb{R}$ and $a_{n+1} = -\frac{1}{2}a_n + 1$.



Characterize the convergence behavior of the sequence $\{a_n\}$ for each choice of initial value a_1.

Solution. The sequence converges to $2/3$ for every initial value. Note that the sequence in this case in not monotonic.

PROBLEM 5. Summarize the convergence behavior of a sequence defined by $a_1 \in \mathbb{R}$ and $a_{n+1} = ma_n + b$ for $m, b \in \mathbb{R}$.

(a) Consider the cases $|m| < 1$, $|m| > 1$, $m = -1$, and $m = 1$ separately. When does the sequence converge for all initial values? When does the sequence converge for only a special initial value (in which case the sequence is constant)?

(b) When is the sequence monotone?

(c) How does $|m|$ affect the rate of convergence or divergence?

Solution.

(a) When $|m| < 1$, the sequence converges for all initial values. It is constant when the initial value corresponds to the point in which $y = mx + b$ meets the line $y = x$, i.e., when $a_1 = ma_1 + b$. This happens when $a_1 = b/(1 - m)$.

When $|m| > 1$ the sequence diverges for all initial values except when $a_1 = b/(1-m)$. With this initial value, we get a constant sequence, as discussed above.

Next, consider the case $m = 1$. So the recursion is $a_{n+1} = a_n + b$ and the sequence is:

$$a_1, a_1 + b, a_1 + 2b, a_1 + 3b, \ldots$$

which converges if and only if $b = 0$, in which case we get a constant sequence.

Finally, consider the case where $m = -1$. The sequence in that case is

$$a_1,\ -a_1 + b,\ a_1,\ -a_1 + b,\ a_1,\ \ldots$$

Convergence occurs exactly when the initial value is $b/2$, which is $b/(1 - m)$ for the case $m = -1$.

(b) The sequence is monotone when $m > 0$ or in those special cases in which it is constant.

(c) The rate of convergence or divergence decreases as $|m|$ gets close to 1 (for generic initial conditions).

PROBLEM 6. (Extra time.) Let $a_1 = 1$, and for $n \geq 1$, let $a_{n+1} := \frac{1}{2}a_n + \frac{1}{3}$. From the data shown below, it looks like the sequence $\{a_n\}$ is monotone decreasing and converging to $2/3$.

$$\begin{array}{llll} a_1 = 1.000000\ldots, & a_2 = 0.833333\ldots, & a_3 = 0.750000\ldots, & a_4 = 0.708333\ldots, \\ a_5 = 0.687500\ldots, & a_6 = 0.677083\ldots, & a_7 = 0.671875\ldots, & a_8 = 0.669271\ldots, \\ \quad\ldots\quad, & a_{20} = 0.666667\quad, & \quad\ldots & \end{array}$$

(a) Prove that $\{a_n\}$ is bounded below by $2/3$.

(b) Prove that $\{a_n\}$ is monotone decreasing.

(c) Thus, by the MCT, the sequence converges. Find its limit.

Solution.

(a) We will prove this by induction. For the base case, we have $a_1 = 1 \geq 2/3$. Suppose that $a_n \geq 2/3$ for some $n \geq 1$. Then

$$a_{n+1} = \frac{1}{2}a_n + \frac{1}{3} \geq \frac{1}{2} \cdot \frac{2}{3} + \frac{1}{3} = \frac{2}{3}.$$

The result follows by induction.

(b) We will prove this by induction. For the base case, we have $a_1 = 1 \geq 5/6 = a_2$. Suppose that $a_n \geq a_{n+1}$ for some $n \geq 1$. Then

$$a_{n+1} = \frac{1}{2}a_n + \frac{1}{3} \geq \frac{1}{2}a_{n+1} + \frac{1}{3} =: a_{n+2}.$$

The result follows by induction.

(c) Say $\lim_{n\to\infty} a_n = a$. Then

$$\begin{aligned} a_{n+1} = \frac{1}{2}a_n + \frac{1}{3} \quad &\Rightarrow \quad \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(\frac{1}{2}a_n + \frac{1}{3}\right) \\ &\Rightarrow \quad a = \frac{1}{2}a + \frac{1}{3} \\ &\Rightarrow \quad a = \frac{2}{3}. \end{aligned}$$

PROBLEM 1. Use the squeeze theorem to prove that $\left\{\frac{\sin(n)+\cos(n)}{n^2}\right\}$ converges.

Proof. By the triangle inequality, we have $|\sin(n)+\cos(n)| \leq |\sin(n)|+|\cos(n)| \leq 2$. Therefore,
$$-\frac{2}{n^2} \leq \frac{\sin(n)+\cos(n)}{n^2} \leq \frac{2}{n^2}.$$
Since $\lim_{n\to\infty}(-2/n^2) = \lim_{n\to\infty}(2/n^2) = 0$, the squeeze theorem yields:
$$\lim_{n\to\infty} \frac{\sin(n)+\cos(n)}{n^2} = 0.$$
$\square$

PROBLEM 2. Prove that the sequence $\left\{\cos\left(\frac{n\pi}{3}\right)\right\}$ diverges.

Proof. It has the constant sequences $\{1\}$ and $\{-1\}$ as subsequences, and these subsequences have different limits. (Note that $\cos(6n\pi/3) = 1$ and $\cos\left(6n+3)\pi/3\right)) = -1$ for all n.) $\square$

PROBLEM 3. Given an B-N argument that $\left\{\left(\frac{4}{3}\right)^n\right\}$ diverges to infinity.

Solution. Let $B \in \mathbb{R}_{>0}$.[1] Then
$$\begin{aligned}\left(\frac{4}{3}\right)^n > B &\iff \log\left(\frac{4}{3}\right)^n > \log(B) &&\text{(since } \log(x)) \text{ is an increasing function)}\\ &\iff n\log\left(\frac{4}{3}\right) > \log(B)\\ &\iff n > \log(B)/\log\left(\frac{4}{3}\right) &&\text{(since } \log(4/3) > 0)\end{aligned}$$

Therefore, if $n > N := \log(B)/\log\left(\frac{4}{3}\right)$, we have that
$$\left(\frac{4}{3}\right)^n > B,$$
as desired.

PROBLEM 4. In this problem, we will show that a complex sequence converges if and only if the sequences of its real and complex parts converge. Let $\{z_n\}$ be a complex sequence where $z_n = a_n + ib_n$ for all n. You will only need to use an ε-N proof for part (b).

(a) Show that if $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then $\lim_{n\to\infty} z_n = a+bi$.

[1] It is OK to assume that $B > 0$. If you give me $B' \leq 0$ and I make $a_n > B$, then it follows automatically that $a_n > B'$.

(b) If $\lim_{n\to\infty} z_n = z$, show that $\lim_{n\to\infty} \overline{z_n} = \overline{z}$. (Use standard properties of conjugation and the modulus.)

(c) Use the result in (b) to show that if $\lim_{n\to\infty} z_n = a + bi$, then $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$.

Solution.

(a) This result follows directly from our limit theorems:

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} (a_n + ib_n) = \lim_{n\to\infty} a_n + i \lim_{n\to\infty} b_n = a + ib.$$

(b) Given $\varepsilon > 0$, since $\lim_{n\to\infty} z_n = z$, there exists N such that $n > N$ implies $|z - z_n| < \varepsilon$. It follows that for $n > N$,

$$|\overline{z} - \overline{z_n}| = |\overline{z - z_n}| = |z - z_n| < \varepsilon.$$

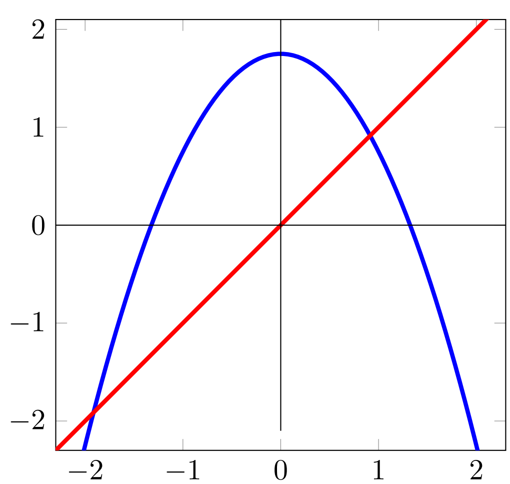
(c) Using our limit theorems we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{z_n + \overline{z_n}}{2} \right) = \frac{\lim_{n\to\infty}(z_n + \overline{z_n})}{2} = \frac{z + \overline{z}}{2} = a.$$

Similarly,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \left(\frac{z_n - \overline{z_n}}{2i} \right) = \frac{\lim_{n\to\infty}(z_n - \overline{z_n})}{2i} = \frac{z - \overline{z}}{2} = b.$$

PROBLEM 5. Here is a diagram for the sequence $a_1 \in \mathbb{R}$ and $a_{n+1} = 1.75 - a_n^2$ for $n > 1$. Recall the method we developed last time for constructing the sequence geometrically: start at some point on the x-axis, then repeatedly draw lines vertically to blue (the graph of $y = 1.75 - x^2$) and horizontally to red (the graph of $y = x$); the resulting sequence of heights (from the x-axis to the blue graph) is $\{a_n\}$. What sequences do you get in this case for various initial points a_1? (One interesting starting point is where the first height $a_1 = -0.5$.)



Solution. See http://csc.ucdavis.edu/~chaos/courses/poci/Readings/ch2.pdf, for instance. In general, look on the internet for "discrete dynamical systems" and "bifurcation diagrams".

Starting with $a_1 = -1/2$, we get $a_2 = 7/4 - (1/2)^2 = 3/2$, $a_3 = 7/4 - (3/2)^2 = -1/2$, etc.:

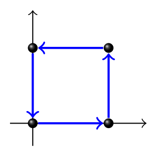
$$-\frac{1}{2},\ \frac{3}{2},\ \frac{1}{2},\ -\frac{1}{2},\ \frac{3}{2},\ldots$$

PROBLEM 1. In what sense is $\sum_{n=0}^{\infty} i^n$ a sequence? Draw this sequence in the complex plane.

Solution. The series $\sum_{n=0}^{\infty} i^n$ is the sequence of partial sums:

$$s_0 = 1,\ s_1 = 1 + i,\ s_2 = 1 + i + i^2 = 1 + i - 1 = i,\ s_3 = s_2 + i^3 = 0,$$

and $s_n = s_{n-4}$ for $n \geq 4$. This sequence is depicted below:



PROBLEM 2. Let $\{a_n\}$ be a sequence of real numbers.

(a) Critique the statement that $\sum_{n=0}^{\infty} a_n$ is convergent if and only if its sequence of partial sums is bounded. Give a proof or a counterexample for both implications.

(b) Does anything change if $\{a_n\}$ is a sequence of *nonnegative* real numbers?

Solution.

(a) For the series $\sum_{n=0}^{\infty} a_n$ to be convergent means that its sequence of partial sums is convergent. A convergent sequence is bounded. Hence, if $\sum_{n=0}^{\infty} a_n$ converges, its sequence of partial sums is bounded. For the converse, consider the series $\sum_{n=0}^{\infty}(-1)^n$. Its sequence of partial sums is $1, 0, 1, 0, \ldots$ and hence is bounded. However, it does not converge.

(b) If $a_n \geq 0$ for all n, then the sequence of partial sums for $\sum_{n=0}^{\infty} a_n$ is monotonically increasing. By the monotone convergence theorem, then, the series converges if and only if its sequence of partial sums is bounded above.

PROBLEM 3. Determine whether the following series converge, and in the case one does, find its sum. If the sum is complex, express the answer in the form $a + bi$ with $a, b \in \mathbb{R}$.

$$(a)\ \sum_{n=2}^{\infty}(-1)^n \frac{3^{2n+2}}{10^n} \qquad \text{(b)}\ \sum_{n=0}^{\infty}\left(\frac{2+i}{2}\right)^n \qquad \text{(c)}\ \sum_{n=0}^{\infty}\left(\frac{3+i}{5}\right)^n.$$

Solution.

(a) We have

$$\sum_{n=2}^{\infty}(-1)^n \frac{3^{2n+2}}{10^n} = \sum_{n=2}^{\infty}(-1)^n \frac{3^{2n} \cdot 3^2}{10^n} = \sum_{n=2}^{\infty} 9 \cdot \left(-\frac{9}{10}\right)^n.$$

Since $|-9/10| < 1$, the series converge, and its value is

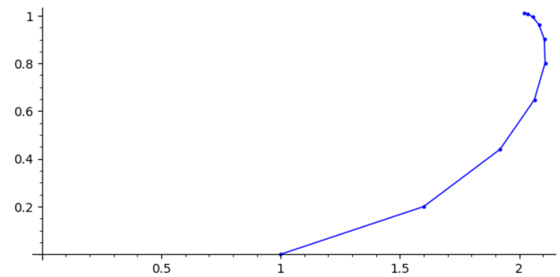
$$\sum_{n=2}^{\infty} 9 \cdot \left(-\frac{9}{10}\right)^n. = 9 \cdot \left(\frac{9}{10}\right)^2 \cdot \frac{1}{1-(-9/10)} = 9 \cdot \left(\frac{9}{10}\right)^2 \cdot \frac{10}{19} = \frac{729}{190}.$$

(b) Since $|(3+i)/2| = \sqrt{10}/2 > 1$, there series diverges since its a geometric series with ratio greater than 1.

(c) Here, $|(3+i)/5| = \sqrt{10}/5 < 1$, so this geometric series is summable. The value is

$$\frac{1}{1-\frac{3+i}{5}} = \frac{5}{2-i} = \frac{5}{2-i}\cdot\frac{2+i}{(2+i)} = \frac{5(2+i)}{5} = 2+i.$$

Here is a picture of the convergence of the sequence of partial sums:



PROBLEM 4. Express $0.99999\ldots$ as a geometric series, and sum the series. Do the same for $6.232323\ldots$ to express this number as a quotient of integers.

Solution. We have

$$\begin{aligned}
0.999\ldots &= \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \ldots \\
&= \sum_{n=1}^{\infty} 9\cdot\left(\frac{1}{10}\right)^n \\
&= 9\cdot\left(\frac{1}{10}\right)\sum_{n=0}^{\infty}\cdot\left(\frac{1}{10}\right)^n \\
&= 9\cdot\left(\frac{1}{10}\right)\frac{1}{1-1/10} \\
&= 9\cdot\left(\frac{1}{10}\right)\frac{10}{9} \\
&= 1.
\end{aligned}$$

Similarly,

$$\begin{aligned}
6.232323\ldots &= 6 + \frac{23}{100} + \frac{23}{100^2} + \frac{23}{100^3} + \ldots \\
&= 6 + \sum_{n=1}^{\infty} 23\cdot\left(\frac{1}{100}\right)^n
\end{aligned}$$

$$= 6 + 23 \cdot \left(\frac{1}{100}\right) \sum_{n=0}^{\infty} \cdot \left(\frac{1}{100}\right)^n$$

$$= 6 + 23 \cdot \left(\frac{1}{100}\right) \frac{1}{1 - 1/100}$$

$$= 6 + 23 \cdot \left(\frac{1}{100}\right) \frac{100}{99}$$

$$= 6 + \frac{23}{99}$$

$$= \frac{617}{99}.$$

PROBLEM 5. Sum the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$.

Solution. Using a variant of the telescoping sum argument given in the notes, we get

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$= \frac{1}{2}\left(\left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots\right)$$

$$= \frac{1}{2}\left(1 + \frac{1}{2}\right)$$

$$= \frac{3}{4}.$$

PROBLEM 1. What does the n-th term test say about the following series?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{ni^n}{n+1}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n) + i\sin(n)}{n}$ (c) $\displaystyle\sum_{n=2}^{\infty} \frac{n}{\log(n)}$.

Solution.

(a) Since $\lim_{n\to\infty} \frac{n}{n+1}$ exists and $\lim_{n\to\infty} i^n$ does not exist, it follows that the limit of the n-th term of the series does not exist. Therefore, the n-term test says the series does not converge.

(b) The limit of the n-term of the series is 0. To see that, given $\varepsilon > 0$, let $N = 1/\varepsilon$. Then if $n > N$, it follows that

$$\left|\frac{\cos(n) + i\sin(n)}{n}\right| \leq \frac{1}{n} < \frac{1}{N} = \varepsilon.$$

Another way to see that the limit of the n-term is 0 is to use the fact that $\lim_{n\to\infty} b_n = 0$ if and only if $\lim_{n\to\infty} |b_n| = 0$ and then apply the squeeze theorem to

$$0 \leq \left|\frac{\cos(n) + i\sin(n)}{n}\right| = \frac{|\cos(n) + i\sin(n)|}{n} \leq \frac{1}{n}.$$

Since the limit of the n-term exists, the n-term test is inconclusive.

(c) Since $n > \log(n)$ for $n \geq 1$, we have that $\lim_{n\to\infty} \frac{n}{\log(n)} \neq 0$. (In fact, the sequence diverges to ∞.) Since the limit of the n-th term is not zero, the series diverges.

PROBLEM 2. Let $p \in \mathbb{R}$. For the following, you may use the fact that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. *Use the comparison test* to say whether the following series converge or diverge:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ (b) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{\sqrt{n-2}}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{5^n(n+2)}$.

Solution.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ converges by comparison with the convergent p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ since

$$0 \leq \frac{1}{n^3+1} \leq \frac{1}{n^3}.$$

(b) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{\sqrt{n-2}}$ diverges by comparison with divergent p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ since

$$0 \leq \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{n-2}}$$

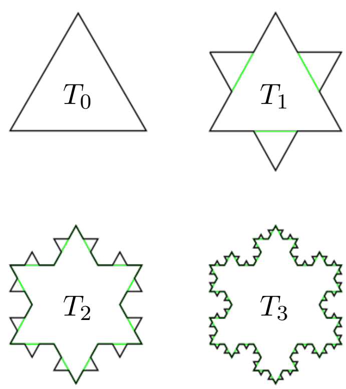
for all $n \geq 3$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{5^n(n+2)}$ converges by comparison with the convergent geometric series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n$ since

$$0 \leq \frac{4^n}{5^n(n+2)} \leq \frac{4^n}{5^n} = \left(\frac{4}{5}\right)^n.$$

Problem 3. Let T_0 be an equilateral triangle of side length 1. Recursively define T_n for $n \geq 1$ by replacing each side s by four line segments of size equal to a third of that of s like so:

Thus, we have



Define T to be $\lim_{n\to\infty} T_n$ (which you can imagine even though we have not defined what we mean by the limit of a geometric shape). Find the perimeter and area of T_n for each each $n \geq 1$. (You should see that T has infinite perimeter but finite area.)

Hints: (i) Moving from T_{n-1} to T_n, by what factor does the number of edges increase (each hold edge becomes how many new edges)? By what factor does the length get scaled? This should allow you to compute the perimeter.

(ii) When computing the area, note that when moving to the next triangle, you add a number of triangles equal to the number of line segments in the previous triangle. That number comes from (i). The new triangles with have an area equal to 1/9-th the triangles added in the previous step (why?).

Solution. We start with T_0 which has 3 segments, each of length 1. At each step the number of segments is quadrupled and the length is divided by three. So

$$\text{perimeter}(T_0) = 3, \text{perimeter}(T_1) = 3 \cdot 4 \cdot \frac{1}{3}, \text{perimeter}(T_2) = 3 \cdot 4^2 \cdot \frac{1}{3^2}, \text{perimeter}(T_3) = 3 \cdot 4^3 \cdot \frac{1}{3^3}.$$

In general, the perimeter is given

$$\text{perimeter}(T_n) = 3\left(\frac{4}{3}\right)^n,$$

which diverges as $n \to \infty$.

Next, consider the areas. At each step we add a number of triangles equal to the number of line segments of the previous step. For our earlier reasoning, the number of line segments in T_n is $3 \cdot 4^{n-1}$. Each of the added triangles has 1/9-th the area of the triangles added at the previous step. Thus,

$$\text{area}(T) = \frac{\sqrt{3}}{4}\left(1 + 3 \cdot \frac{1}{9} + (3 \cdot 4)\frac{1}{9^2} + (3 \cdot 4^2)\frac{1}{9^3} + \cdots\right)$$

s

$$\begin{aligned}
&= \frac{\sqrt{3}}{4}\left(1 + 3\sum_{n=1}^{\infty} \frac{4^{n-1}}{9^n}\right) \\
&= \frac{\sqrt{3}}{4}\left(1 + \frac{3}{4}\sum_{n=1}^{\infty} \left(\frac{4}{9}\right)^n\right) \\
&= \frac{\sqrt{3}}{4}\left(1 + \left(\frac{3}{4}\right)\left(\frac{4}{9}\right)\sum_{n=0}^{\infty} \left(\frac{4}{9}\right)^n\right) \\
&= \frac{\sqrt{3}}{4}\left(1 + \frac{1}{3} \cdot \frac{1}{1 - 4/9}\right) \\
&= \frac{2\sqrt{3}}{5}.
\end{aligned}$$

PROBLEM 1. Use the limit comparison test to determine whether the following series converge. You may use the fact that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{5n^2 - 6n + 3}{4n^6 + n^3 + 7} \qquad\qquad \text{(b)} \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+2}}$$

Solution.

(a) This series converges by limit comparison with the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ since, as $n \to \infty$,

$$\left(\frac{5n^2 - 6n + 3}{4n^6 + n^3 + 7}\right) \Big/ \left(\frac{1}{n^4}\right) = \frac{5n^6 - 6n^5 + 3n^4}{4n^6 + n^3 + 7} \longrightarrow \frac{5}{4} \neq 0.$$

(b) This series diverges by limit comparison with the divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$ since, as $n \to \infty$,

$$\left(\frac{1}{\sqrt{n^2+2}}\right) \Big/ \left(\frac{1}{n}\right) = \frac{n}{\sqrt{n^2+2}} = \frac{\frac{1}{n} \cdot n}{\frac{1}{n} \cdot \sqrt{n^2+2}} = \frac{1}{\sqrt{1 + \frac{2}{n^2}}} \longrightarrow 1 \neq 0.$$

PROBLEM 2. Are the following series absolutely convergent, conditionally convergent, or divergent?

$$\text{(a)} \quad \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{3n+1} \qquad \text{(b)} \quad \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}}{n+4} \qquad \text{(c)} \quad \sum_{n=0}^{\infty} \frac{(-3)^n}{5^{n+1}}.$$

Solution.

(a) This series is divergent by the n-th term test since its sequence of terms diverges. In particular,

$$\lim_{n\to\infty} (-1)^{n+1} \frac{n}{3n+1} \neq 0.$$

(b) This series is conditionally convergent. To apply the alternating series test, we first check that $\{\sqrt{n}/(n+4)\}$ is (eventually) decreasing by showing the derivative with respect to n is negative. Using the quotient rule,

$$\left(\frac{n^{1/2}}{n+4}\right)' = \frac{\frac{1}{2}n^{-1/2}(n+4) - n^{1/2}}{(n+4)^2} = \frac{(n+4) - 2n}{2\sqrt{n}(n+4)^2} = \frac{-n+4}{2\sqrt{n}(n+4)^2} < 0$$

for $n > 4$. Next, notice that $\lim_{n\to\infty} \frac{\sqrt{n}}{n+4} = 0$. (To give a formal proof of this fact, we can use the squeeze theorem since $0 \leq \frac{\sqrt{n}}{n+4} \leq \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$, and we know that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.)

We have just shown that the series is convergent. It is conditionally convergent since $\sum_{n=1}^{\infty} \frac{n}{n+4}$ diverges by limit comparison with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$: as $n \to \infty$,

$$\left(\frac{\sqrt{n}}{n+4}\right) \Big/ \left(\frac{1}{\sqrt{n}}\right) = \frac{n}{n+4} \longrightarrow 1 \neq 0.$$

(c) This series is absolutely convergent since it is essentially a geometric series with ratio less than 1:

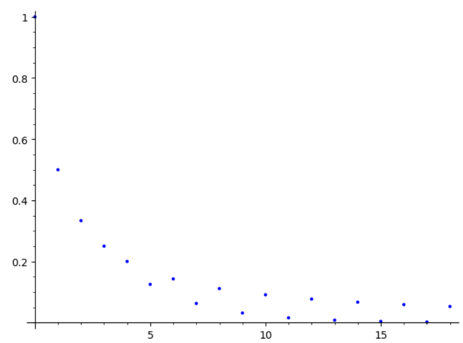
$$\sum_{n=0}^{\infty} \frac{3^n}{5^{n+1}} = \frac{1}{5} \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n$$

and $3/5 < 1$.

PROBLEM 3. What does the alternating series test say about the following series?

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{2^2} + \frac{1}{5} - \frac{1}{2^3} + \frac{1}{7} - \frac{1}{2^4} + \cdots$$

Here is a plot of the first few partial sums, $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{2^2}, \frac{1}{5}, \ldots$:



Solution. The alternating series test is inconclusive since the terms of the series are not monotonically decreasing.

PROBLEM 4. Consider the series from the previous problem:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{2^2} + \frac{1}{5} - \frac{1}{2^3} + \frac{1}{7} - \frac{1}{2^4} + \cdots. \tag{1}$$

Here is a typical partial sum:

$$\begin{aligned} s_{2k+1} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{2^2} + \cdots - \frac{1}{2^k} + \frac{1}{2k+1} \\ &= 1 + \frac{1}{3} + \cdots + \frac{1}{2k+1} - \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k}\right). \end{aligned}$$

(a) Prove that $\sum_{k=0}^{\infty} \frac{1}{2k+1}$ diverges to infinity.

(b) Find a lower bound for s_{2k+1} that allows you to show that the series (1) diverges.

(c) Why doesn't this example violate the alternating series test?

Solution.

(a) The series diverges by limit comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$ since, as $k \to \infty$,

$$\left(\frac{1}{2k+1}\right) \Big/ \left(\frac{1}{k}\right) = \frac{k}{2k+1} \longrightarrow \frac{1}{2} \neq 0.$$

Since the terms of the series are positive, the partial sums for the series are monotonically increasing. Therefore, by the monotone convergence theorem, the series is not bounded. Thus, the series diverges to infinity.

(b) We have

$$\frac{1}{2} + \cdots + \frac{1}{2^k} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2}\left(\frac{1}{1-1/2}\right) = 1.$$

Therefore,

$$s_{2k+1} = 1 + \frac{1}{3} + \cdots + \frac{1}{2k+1} - \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k}\right) \geq \left(1 + \frac{1}{3} + \cdots + \frac{1}{2k+1}\right) - 1$$

Since $\sum_{k=1}^{\infty} \frac{1}{2k+1}$ diverges to infinity, the series (1) diverges.

(c) As stated in the previous problem, the alternating series does not apply here since the term of the series are not monotonically decreasing.

PROBLEM 1. Apply the ratio test to each of the following series, and state what conclusion may be drawn:

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{n!}{5^n} \qquad \text{(b)} \quad \sum_{n=1}^{\infty} \frac{n^2}{(2n)!} \qquad \text{(c)} \quad \sum_{n=1}^{\infty} \frac{1}{2n^2} \qquad \text{(d)} \quad \sum_{n=1}^{\infty} \frac{n!}{n^n}$$

For part (d), you may use the fact that $\lim_{n\to\infty}(1+1/n)^n = e$.

SOLUTION:

(a) We have

$$\frac{\frac{(n+1)!}{5^{n+1}}}{\frac{n!}{5^n}} = \frac{(n+1)!}{n!} \cdot \frac{5^n}{5^{n+1}} = \frac{n+1}{5} \longrightarrow \infty.$$

Hence, the series diverges by the ratio test.

(b) We have

$$\frac{\frac{(n+1)^2}{(2(n+1))!}}{\frac{n^2}{(2n)!}} = \frac{(n+1)^2}{n^2} \cdot \frac{(2n)!}{(2n+2)!} = \left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{(2n+2)(2n+1)} \longrightarrow 0.$$

Hence, the series converges by the ratio test.

(c) We have

$$\frac{\frac{1}{2(n+1)^2}}{\frac{1}{2n^2}} = \left(\frac{n+1}{n}\right)^2 \longrightarrow 1.$$

So the ratio test is inconclusive.

(d) We have

$$\begin{aligned}
\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} &= \frac{n^n}{(n+1)^{n+1}} \cdot \frac{(n+1)!}{n!} \\
&= \frac{n^n}{(n+1)^{n+1}} \cdot (n+1) \\
&= \frac{n^n}{(n+1)^n} \\
&= \frac{1}{(1+\frac{1}{n})^n} \\
&= \frac{1}{e} < 1.
\end{aligned}$$

Hence, the series converges by the ratio test.

PROBLEM 2. Apply the integral test to each of the following series, and state what conclusion may be drawn:

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \qquad \text{(b)} \quad \sum_{n=1}^{\infty} \frac{1}{n^{4/3}} \qquad \text{(c)} \quad \sum_{n=1}^{\infty} \frac{n^2}{e^{n^3}}$$

SOLUTION:

(a) We have

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{n\to\infty} \int_1^n x^{-1/2}\,dx = \lim_{n\to\infty} 2x^{1/2}\big|_1^n = 2\lim_{n\to\infty} \left(\sqrt{n}-1\right) = \infty.$$

Hence, the series diverges.

(b) We have

$$\int_1^\infty \frac{1}{x^{4/3}} = \lim_{n\to\infty} \int_1^n x^{-4/3}\,dx = -3\lim_{n\to\infty} x^{-1/3}\big|_1^n = -3\lim_{n\to\infty} \left(\frac{1}{\sqrt[3]{n}} - 1\right) = 3.$$

Since the integral converges, so does the series.

(c) We have

$$\int_1^\infty \frac{x^2}{e^{x^3}}\,dx = \lim_{n\to\infty} \int_1^n x^2 e^{-x^3}\,dx = -\frac{1}{3}\lim_{n\to\infty} e^{-x^3}\big|_1^n = -\frac{1}{3}\lim_{n\to\infty} \left(\frac{1}{e^{n^3}} - \frac{1}{e}\right) = \frac{1}{3e}.$$

Since the integral converges, so does the series.

PROBLEM 3. As a consequence of our limit theorems, we know that if $\sum_n a_n$ and $\sum_n b_n$ converge, then so do $\sum_n (a_n + b_n)$ and $\sum_n ca_n$ for all constants c. It turns out that it is not necessarily true that $\sum_n a_n b_n$ converges. As a special case (where $a_n = b_n$), find a series $\sum_n a_n$ such that $\sum_n a_n = 0$, and yet $\sum_n a_n^2$ diverges to ∞.

SOLUTION: Let $\{a_n\}$ be the sequence

$$1, -1, \sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{3}}, -\sqrt{\frac{1}{3}}, \ldots.$$

The sequence of partial sums i

$$1, 0, \sqrt{\frac{1}{2}}, 0, \sqrt{\frac{1}{3}}, 0, \ldots,$$

which converges to 0.

On the other hand, the sequence $\left\{a_n^2\right\}$ is the sequence

$$1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \ldots,$$

which diverges by comparison with the harmonic series.

For convenience:

$\lim_{x\to a} f(x) = L$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x-a| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

PROBLEM 1. Find $\lim\limits_{x\to 9} x^2$, and provide an ε-δ proof.

SOLUTION: Claim: $\lim_{x\to 9} x^2 = 81$.

Proof. Given $\varepsilon > 0$, let $\delta = \min\{1, \varepsilon/19\}$ and suppose that $0 < |x-9| < \delta$. Then, since $\delta \leq 1$, it follows that $8 < x < 10$, and hence $17 < x+9 < 19$. Combining this with the fact that $\delta \leq \varepsilon/19$, we have

$$|x^2 - 81| = |x+9||x-9| < 19|x-9| < 19\delta \leq 19\frac{\varepsilon}{19} = \varepsilon.$$

$\square$

PROBLEM 2. Find $\lim\limits_{x\to 3} \dfrac{1}{2+x}$, and provide an ε-δ proof.

SOLUTION: Claim: $\lim\limits_{x\to 3} \dfrac{1}{2+x} = \dfrac{1}{5}$.

Proof. Given $\varepsilon > 0$, let $\delta = \min\{1, 20\varepsilon\}$ and suppose that $0 < |x-3| < \delta$. Since $\delta \leq 1$, we have $2 < x < 4$, and hence $20 < 5(2+x) < 30$. Combining this with the fact that $\delta \leq 20\varepsilon$, we have

$$\left|\frac{1}{2+x} - \frac{1}{5}\right| = \left|\frac{5-(2+x)}{5(2+x)}\right| = \left|\frac{3-x}{5(2+x)}\right| = \frac{|x-3|}{|5(2+x)|} < \frac{1}{20}|x-3| < \frac{1}{20}\delta \leq \varepsilon.$$

$\square$

PROBLEM 3. Find $\lim\limits_{x\to 1} \left(x^2 + 3x + 2\right)$, and provide an ε-δ proof.

SOLUTION: Claim: $\lim_{x\to 1} \left(x^2 + 3x + 2\right) = 6$.

Proof. Given $\varepsilon > 0$, let $\delta = \min\{1, \varepsilon/6\}$ and suppose that $0 < |x-1| < \delta$. Then since $\delta \leq 1$, we have $0 < x < 2$, and hence, $4 < x+4 < 6$. Combining this with the fact that $\delta \leq \varepsilon/6$, we have

$$\begin{aligned} |x^2 + 3x + 2 - 6| &= |x^2 + 3x - 4| \\ &= |(x+4)(x-1)| \\ &= |x+4||x-1| \\ &< 6|x-1| \\ &< 6\delta \leq 6 \cdot \frac{\varepsilon}{6} = \varepsilon. \end{aligned}$$

□

Problem 4. Define

$$f\colon \mathbb{R} \to \mathbb{R}$$
$$x \mapsto \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational.} \end{cases}$$

Does $\lim_{x\to 0} f(x)$ exist? If so, then provide an ε-δ proof. If not, then provide an ε that can't be beat by any δ.

Solution: For sake of contradiction, suppose that $\lim_{x\to 0} f(x) = L$ for some $L \in \mathbb{R}$, and let $\varepsilon = 1$. Then we can find $\delta > 0$ such that $0 < |x| < \delta$ implies $|f(x) - L| < \varepsilon = 1$. There exist both a rational number $p \neq 0$ and an irrational number $q \neq 0$ within a distance of δ from 0. (For instance, we could let $p = 1/2^n$ and $q = \sqrt{2}/2^n$ for a suitably large n.) Then

$$2 = |f(p) - f(q)| = |(f(p) - L) - (f(q) - L)| \leq |f(p) - L| + |f(q) - L| < 1 + 1 = 2,$$

a contradiction.

PROBLEM 1. Let $f\colon \mathbb{C} \to \mathbb{C}$ be given by $f(z) = 3z^2 + 2$. Compute $f'(3i)$ directly from the definition of the derivative.

SOLUTION: We have

$$\begin{aligned}
\lim_{x \to a} f'(3i) &= \lim_{h \to 0} \frac{f(3i+h) - f(3i)}{h} \\
&= \lim_{h \to 0} \frac{(3(3i+h)^2 + 2) - (3(3i)^2 + 2)}{h} \\
&= \lim_{h \to 0} \frac{(3(-9 + 6ih + h^2) + 2) - (3(-9) + 2)}{h} \\
&= \lim_{h \to 0} \frac{18ih + 3h^2}{h} \\
&= \lim_{h \to 0} (18i + 3h) \\
&= 18i.
\end{aligned}$$

PROBLEM 2. Let $A, B, C \subseteq F$ where $F = \mathbb{R}$ or $\mathbb{C}$, and suppose that $f\colon A \to B$ and $g\colon B \to C$ are continuous functions. Show that $g \circ f$ is continuous by filling in the blanks below.

Proof. Let $a \in A$, and let $\varepsilon > 0$. Since g is continuous at $f(a)$, there exists $\delta > 0$ such that

(1) $$|x - f(a)| < \delta \quad \Rightarrow \quad \boxed{}\,.$$

Fix this δ. Since f is continuous at a, there exists $\eta > 0$ such that

(2) $$|x - a| < \eta \quad \Rightarrow \quad \boxed{}$$

Combining (1) and (2), we see that

$$|x - a| < \eta \quad \Rightarrow \quad \boxed{}$$

Thus, $g \circ f$ is continuous at a. $\square$

Proof. Let $\varepsilon > 0$. Since g is continuous at $f(a)$, there exists $\delta > 0$ such that

(3) $$|x - f(a)| < \delta \quad \Rightarrow \quad |g(x) - g(f(a))| < \varepsilon.$$

Fix this δ. Since f is continuous at a, there exists $\eta > 0$ such that

(4) $$|x - a| < \eta \quad \Rightarrow \quad |f(x) - f(a)| < \delta.$$

Combining (3) and (4), we see that

$$|x-a|<\eta \quad \Rightarrow \quad |f(x)-f(a)|<\delta \quad \Rightarrow \quad |g(f(x))-g(f(a))|<\varepsilon.$$

Thus, $g \circ f$ is continuous at a. ☐

PROBLEM 3.

(a) Let $z, w \in \mathbb{C}$. What do the triangle inequality and the reverse triangle inequality say about $|z+w|$? What about $|z-w|$?

(b) Prove that the function $f\colon \mathbb{C} \to \mathbb{C}$ defined by $f(x)=|x|$ is continuous.

SOLUTION:

(a) Let $z, w \in \mathbb{C}$. The triangle inequality says that

$$|z+w| \leq |z|+|w|.$$

Replacing w by $-w$ in the above inequality yields

$$|z-w|=|z+(-w)| \leq |z|+|-w|=|z|+|w|.$$

Hence,

$$|z-w| \leq |z|+|w|.$$

The reverse triangle inequality says that

$$|z+w| \geq ||z|+|w||,$$

It follows that

$$|z-w|=|z+(-w)| \geq ||z|+|-w||=||z|+|w||,$$

i.e.,

$$|z-w| \geq ||z|+|w||.$$

(b) *Proof.* Let $a \in \mathbb{C}$. Given $\varepsilon>0$, let $\delta=\varepsilon$, and suppose that $|x-a|<\delta$. Then, by the reverse triangle inequality,

$$|f(x)-f(a)|=||x|-|a|| \leq |x-a|<\delta=\varepsilon.$$

☐

PROBLEM 1. Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} z^n$.

Solution. Using the power series ratio test, we find

$$\begin{aligned} \lim_{n\to\infty} \left(\frac{(2n)!}{(n!)^2}\right) \Big/ \left(\frac{(2(n+1))!}{((n+1)!)^2}\right) &= \lim_{n\to\infty} \left(\frac{(n+1)!}{n!}\right)^2 \cdot \frac{(2n)!}{(2n+2)!} \\ &= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} \\ &= \frac{1}{4}. \end{aligned}$$

Therefore, $R = \frac{1}{4}$.

PROBLEM 2. Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} z^{2n}$.

Solution. Let $w := z^2$, and consider the power series $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} w^n$. From Problem 1, we know that this latter series converges for $|w| < \frac{1}{4}$ and diverges for $|w| > \frac{1}{4}$. Since $|w| = |z|^2$, this means the radius of convergence for the original series is $R = \sqrt{\frac{1}{4}} = \frac{1}{2}$.

PROBLEM 3. Compute the radius of convergence of $\sum_{n=0}^{\infty} n! z^n$ and of $\sum_{n=0}^{\infty} \frac{z^n}{n!}$.

Solution. Using the power series ratio test, we find

$$\lim_{n\to\infty} \frac{n!}{(n+1)!} = 0$$

and

$$\lim_{n\to\infty} \frac{(n+1)!}{n!} = \infty.$$

Thus, the radius of convergence of the former series is 0 and of the latter is ∞.

PROBLEM 4. Describe the region in the complex plane where the series $\sum_{n=1}^{\infty} \frac{(5z-2)^n}{n^2 4^n}$ converges. (Don't forget to check the boundary of the region.)

Solution. By the power series ratio test we find

$$\lim_{n\to\infty} \frac{1}{n^2 4^n} \Big/ \frac{1}{(n+1)^2 4^{n+1}} = \lim_{n\to\infty} \frac{(n+1)^2 4^{n+1}}{n^2 4^n} = \lim_{n\to\infty} 4\left(\frac{n+1}{n}\right)^2 = 4.$$

So this series converges where $|5z-2|<4$. We have

$$|5z-2|<4 \quad \Leftrightarrow \quad \left|z-\frac{2}{5}\right|<\frac{4}{5}.$$

Thus, the series converges absolutely in the open ball of radius $\frac{4}{5}$ centered at $\frac{2}{5}\in\mathbb{C}$, i.e., at the point $\left(\frac{2}{5},0\right)$.

On the boundary of the disc, where $|5z-2|=4$, the series converges absolutely:

$$\sum_{n=1}^{\infty}\left|\frac{(5z-2)^n}{n^2 4^n}\right|=\sum_{n=1}^{\infty}\frac{4^n}{n^2 4^n}=\sum_{n=1}^{\infty}\frac{1}{n^2},$$

which converges by the p-test.

Problem 5. What is the radius of convergence of the series $f(z)=\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}z^n$. What happens on the boundary of its disc of convergence?

Solution. Apply the power series ratio test:

$$\lim_{n\to\infty}\left|\frac{(-1)^n}{n}\right|\Big/\left|\frac{(-1)^{n+1}}{n+1}\right|=1.$$

Thus, the radius of convergence if $R=1$. What about on the boundary of the disc of radius 1 centered at the origin? We have that $f(1)$ is the alternating harmonic series, and hence converges. On the other hand, $f(-1)$ is the harmonic series, which diverges.

One may show that the series converges at every point on the boundary except for $z=-1$. (For example, use Abel's test as here with $a_n=\frac{1}{n}$ and with $-z$ in place of z.)

Math 112 Group problems, Wednesday Week 12

PROBLEM 1. Consider the geometric series

$$f(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

(a) Compute $zf'(z)$ in two ways and use the result to evaluate $\sum_{n=0}^{\infty} n\left(\frac{2}{3}\right)^n$.

(b) Let $g(z) = zf'(z)$. Thinking of $g(z)$ as both a power series and as a rational function, compute $zg'(z)$ in two ways. Use the result to evaluate $\sum_{n=0}^{\infty} n^2\left(\frac{2}{3}\right)^n$.

Solution.

(a) Taking derivatives, we find

$$f'(z) = \sum_{n=0}^{\infty} nz^{n-1} = \left(\frac{1}{1-z}\right)' = \frac{1}{(1-z)^2}$$

Therefore,

$$zf'(z) = \sum_{n=0}^{\infty} nz^n = \frac{z}{(1-z)^2},$$

and

$$\sum_{n=0}^{\infty} n^2\left(\frac{2}{3}\right)^n = \left(\frac{2}{3}\right) f'\left(\frac{2}{3}\right) = \frac{\left(\frac{2}{3}\right)}{\left(1-\frac{2}{3}\right)^2} = 9 \cdot \frac{2}{3} = 6.$$

(b) Using our previous results, we have

$$g'(z) = \left(\sum_{n=0}^{\infty} nz^n\right)' = \sum_{n=0}^{\infty} n^2 z^{n-1}$$

and

$$\begin{aligned}
g'(z) &= \left(\frac{z}{(1-z)^2}\right)' \\
&= \frac{(z)'(1-z)^2 - z\left((1-z)^2\right)'}{(1-z)^4} && \text{(quotient rule)} \\
&= \frac{(1-z)^2 + 2z(1-z)}{(1-z)^4} \\
&= \frac{(1-z) + 2z}{(1-z)^3} \\
&= \frac{1+z}{(1-z)^3}.
\end{aligned}$$

Therefore,
$$zg'(z) = \sum_{n=0}^{\infty} n^2 z^n = \frac{z(1+z)}{(1-z)^3},$$
and, evaluating at $d = 2/3$,
$$\sum_{n=0}^{\infty} n^2 \left(\frac{2}{3}\right)^n = \frac{\left(\frac{2}{3}\right)\left(1+\frac{2}{3}\right)}{\left(1-\frac{2}{3}\right)^3} = 3^3 \cdot \frac{2}{3} \cdot \frac{5}{3} = 30.$$

Problem 2. Define complex power series by
$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \cdots$$
$$C(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$
$$S(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots.$$
Each has radius of convergence $R = \infty$ (which is easy to check with the ratio test.). Prove the following:
$$E'(z) = E(z), \quad C'(z) = -S(z), \quad \text{and} \quad S'(z) = C(z).$$

Solution. We have
$$E'(z) = \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)' = \sum_{n=0}^{\infty} \frac{nz^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{nz^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = E(z)$$
$$\begin{aligned} C'(z) &= \left(\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}\right)' = \sum_{n=0}^{\infty} (-1)^n \frac{2nz^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty} (-1)^n \frac{2nz^{2n-1}}{(2n)!} \\ &= \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n-1}}{(2n-1)!} = -\sum_{n=0}^{\infty} (-1)^n \frac{z^{2(n+1)-1}}{(2(n-1)-1)!} = -\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \\ &= -S(z) \end{aligned}$$
$$S'(z) = \left(\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}\right)' = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)z^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = C(z).$$

Problem 3. Using the definitions from the previous problem, prove that
$$E(iz) = C(z) + iS(z).$$

Since the series involved are absolutely convergent on $\mathbb{C}$, as far as algebra goes, you can treat them like polynomials, freely rearranging their terms.

Proof.

$$
\begin{aligned}
E(iz) &= \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(iz)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(iz)^{2n+1}}{(2n+1)1!} \\
&= \sum_{n=0}^{\infty} (i)^{2n} \frac{z^{2n}}{(2n)!} + \sum_{n=0}^{\infty} (i)^{2n+1} \frac{z^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} + \sum_{n=0}^{\infty} i(i)^{2n} \frac{z^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \\
&= C(z) + iS(z).
\end{aligned}
$$

□

Problem 4. If there is extra time, try proving that $E(w+z) = E(w)E(z)$. You might first check that the constant terms are the same, then that the order 1 terms are the same, then the order 2 terms, etc. How far can you get? Or you could try proving it all at once. The binomial theorem may be of help:

$$(w+z)^n = \sum_{k=0}^{n} \binom{n}{k} w^{n-k} z^k = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} w^{n-k} z^k.$$

The *binomial coefficients*

$$\binom{n}{0} \quad \binom{n}{1} \quad \binom{n}{2} \quad \cdots \quad \binom{n}{n}$$

form the n-th row of Pascal's triangle. Also, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are power series with radius of convergence R, then for $|z| < R$,

$$f(z)g(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} a_{n-k} b_k \right) z^n,$$

which results from just multiplying out $f(z)g(z)$ as if f and g were polynomials.

Proof. Calculate:

$$
\begin{aligned}
E(w+z) &= \sum_{n=0}^{\infty} \frac{(w+z)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\sum_{k=0}^{n} \binom{n}{k} w^{n-k} z^k}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{w^{n-k} z^k}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} \frac{w^{n-k} z^k}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{(n-k)!k!} w^{n-k} z^k \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{w^{n-k}}{(n-k)!} \frac{z^k}{k!} \\
&= \left(\sum_{n=0}^{\infty} \frac{w^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{z^n}{n!} \right) \\
&= E(w)E(z).
\end{aligned}
$$

□

PROBLEM 1. Let T_n denote the n-th degree Taylor polynomial for $f(x) = x^3 - 3x^2 - x + 3$ centered at $x = 2$.

(a) Fill in the following table, and use it to compute T_n for $n = 1, 2, 3$.

n	$f^{(n)}(x)$	$f^{(n)}(2)$	$f^{(n)}(2)/n!$
0			
1			
2			
3			

(b) Show that $T_3(x) = f(x)$.

(c) Use a computer to plot f, T_1, and T_2 in some interval containing 2 using a different color for each graph. Someone in each group should share their screen so that everyone can view the plot. Here is an example of plotting the same data for the function $\tan(x)$ centered at $x = 0$ using https://sagecell.sagemath.org/:

```
p = plot(tan(x),(x,-1,1),color="black")
q = plot(x+(1/3)*x^3,(x,-1,1),color="blue")
r = plot(x+(1/3)*x^3+(2/15)*x^5,(x,-1,1),color="red")
p+q+r
```

Solution.

(a) We have

n	$f^{(n)}(x)$	$f^{(n)}(2)$	$f^{(n)}(2)/n!$
0	$x^3 - 3x^2 - x + 3$	-6	-3
1	$3x^2 - 6x - 1$	-1	-1
2	$6x - 6$	6	3
3	6	6	1

Therefore,

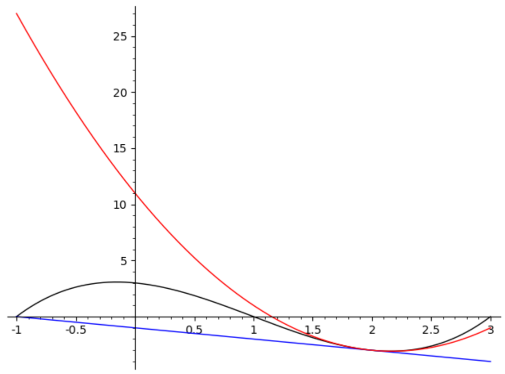
$$\begin{aligned} T_1(x) &= -3 - (x-2) \\ T_2(x) &= -3 - (x-2) + 3(x-2)^2 \\ T_3 &= -3 - (x-2) + 3(x-2)^2 + (x-2)^3. \end{aligned}$$

(b) We have

$$\begin{aligned} T_3 &= -3 - (x-2) + 3(x-2)^2 + (x-2)^3 \\ &= -3 - (x-2) + 3(x^2 - 4x + 4) + (x^3 - 6x^2 + 12x - 8) \\ &= (-3 + 2 + 12 - 8) + (-1 - 12 + 12)x + (3 - 6)x^2 + x^3 \end{aligned}$$

$$= 3 - x - 3x^2 + x^3 = f(x).$$

(c) The plots:



PROBLEM 2. Compute the first-, third-, and fifth-order Taylor polynomials for $f(x) = \sin(x)$ centered at $x = 0$ and use them to approximate $\sin(1)$. Use a computer to see how good these estimates are.

Solution. The Taylor series for $\sin(x)$ centered at 0 is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$$

Truncating the series we find the Taylor polynomials

$$x, \quad x - \frac{x^3}{3!}, \quad \text{and} \quad 1 - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

Our estimates for $\sin(1)$ are

$$1, \quad 1 - \frac{1}{6} = \frac{5}{6} = 0.8333\ldots, \quad \text{and} \quad 1 - \frac{1}{6} + \frac{1}{120} = \frac{101}{120} = 0.8416666\ldots$$

whereas

$$\sin(1) = 0.841470984807897\ldots$$

PROBLEM 3. Consider the function $f(x) = \dfrac{1}{x^2+1}$.

(a) Compute the Taylor series for f centered at $x = 0$. You can do this without calculating derivatives by making an appropriate substitution in the formula for the geometric series

$$\sum_{n=0}^{\infty} y^n = \frac{1}{1-y}.$$

(b) What is the radius of convergence for your series? Given that $f(x)$ is defined for all real numbers, can you think of a reason why its radius of convergence is not $R = \infty$?

Solution.

(a) Letting $y = -x^2$ and substituting gives

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \cdots .$$

(b) The radius of convergence is given by the ratio test:

$$\lim_{n\to\infty} \frac{|(-1)^{n+1}x^{n+1}|}{|(-1)^n x^n|} = \lim_{n\to\infty} |x| = |x|$$

By the ratio test, the radius of convergence is 1.

(c) Thinking about f over the complex numbers, we notice that f blows up at the complex number i, which is a distance of 1 from the origin.

In the following problems, we will be trying to understand the geometry of the complex exponential function

$$\begin{aligned} \mathbb{C} &\to \mathbb{C} \\ z &\mapsto e^z. \end{aligned}$$

Recall that if $z = x + iy \in \mathbb{C}$ for some $x, y \in \mathbb{R}$, we have

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y)).$$

The last expression gives the polar form for e^z: its modulus (length) is e^x and its argument (angle) is y.

PROBLEM 1. What is the image of e^z? Is $z \mapsto e^z$ injective? surjective?

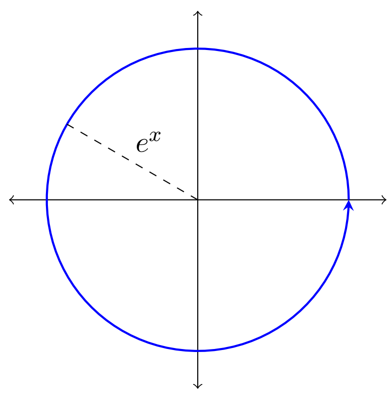
Solution. The image is $\mathbb{C} \setminus \{0\}$.

PROBLEM 2. Describe the image of each vertical line under e^z. It may help to note that a vertical line has the form $\{x + it : t \in \mathbb{R}\}$ for some fixed $x \in \mathbb{R}$. What happens to the images of these lines as $x \to -\infty$? What happens as $x \to \infty$?

Solution. We have

$$e^{x+it} = e^x(\cos(t) + i\sin(t)),$$

which traces out a circle of radius e^x centered at the origin as t varies.



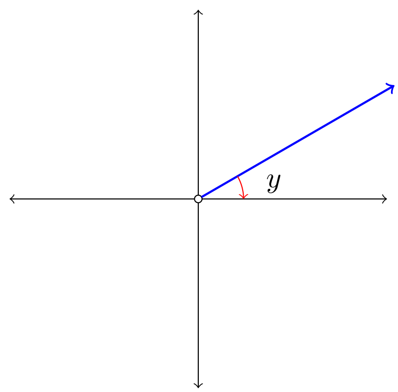
As $x \to -\infty$, the circle's radius approaches 0, and as $x \to \infty$, the circle's radius approached ∞.

PROBLEM 3. Describe the image of each horizontal line under e^z.

Solution. We have

$$e^{t+iy} = e^t(\cos(y) + i\sin(y)),$$

which traces out a ray emanating from the origin with angle y:



Problem 4. (Complex logarithms)

(a) Over the real numbers, we can define the natural logarithm to be the inverse of the exponential function. Why can't we do that with the complex exponential function?

(b) Fix the following horizontal strip of width 2π in the complex plane: $H := \{x + iy \in \mathbb{C} : x \in \mathbb{R} \text{ and } y \in (-\pi, \pi]\}$. Draw two copies of $\mathbb{C}$. In the first one, draw H, and in the second, draw the image K of H under e^z. By thinking about the images of horizontal lines and vertical line segments in H, picture how H is mapped to K by e^z.

(c) Why does $e^z : H \to K$ have an inverse?

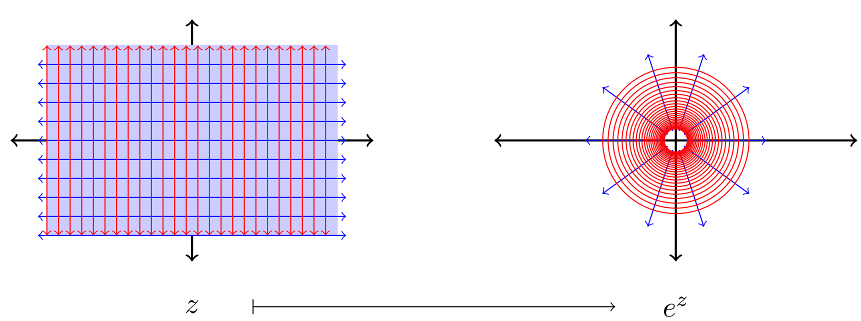
(d) We call this inverse a *branch* of the logarithm and denote it by ln (keeping in mind that this definition depended on fixing a region in the plane on which e^z is injective). Suppose $w \in K$ has polar form $w = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}$. In terms of r and θ, find a formula for $z \in H$ such that $e^z = w$, i.e., find a formula for $\ln(w)$.

(e) For this branch of the logarithm, compute the following: (i) $\ln(1 + i)$, (ii) $\ln(-1)$, and (iii) $\ln(x)$ for $x \in \mathbb{R}$.

Solution.

(a) The function $z \mapsto e^z$ is not injective.

(b) The image is $K = \mathbb{C} \setminus \{0\}$.



(c) The function e^z is injective when restricted to H and then surjective by definition of K.

(d) We have

$$\begin{aligned} \ln\colon \mathbb{C} \setminus \{0\} &\to H \\ re^{i\theta} &\mapsto \ln(r) + i\theta \end{aligned}$$

where $\ln(r)$ is the ordinary natural logarithm for real numbers since $e^{\ln(r)+i\theta} = e^{\ln(r)}e^{i\theta} = re^{i\theta}$. Note that $\theta \in (-\pi, \pi]$.

(e) We have

$$\ln(1+i) = \ln\left(\sqrt{2}e^{i\frac{\pi}{4}}\right) = \ln(\sqrt{2}) + i\frac{\pi}{4}$$

$$\ln(-1) = \ln\left(1 \cdot e^{i\pi}\right) = \ln(1) + i\pi = i\pi$$

$$\ln(x) = \begin{cases} \ln(x) & \text{if } x > 0 \\ \ln(|x|) + i\pi & \text{if } x < 0. \end{cases}$$

where the logs on the right-hand side are the ordinary natural logs for real numbers.

PROBLEM 1. Let $a, b, c \in \mathbb{R}$ and
$$f(x) = x^3 + ax^2 + bx + c.$$
How does the intermediate value theorem guarantee the existence of $\alpha \in \mathbb{R}$ such that $f(\alpha) = 0$? A rigorous proof is not required. Can you generalize this result to other polynomials?

Solution. We have $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$. So there must exist points $u, v \in \mathbb{R}$ such that $f(u) < 0$ and $f(v) > 0$. Since f is continuous, the result now follows from the IVT. The same argument works for any odd-degree polynomial.

PROBLEM 2. Let $f(x) = x^3 - 3x + 1$. Use the intermediate value theorem to prove that the equation $f(x) = 0$ has at least three solutions in $\mathbb{R}$

Solution. We have
$$f(-2) = -1 < 0, \quad f(0) = 1 > 0, \quad f(1) = -1 < 0, \quad f(2) = 3 > 0.$$
The IVT then implies that f has zeros in each of the intervals
$$(-2, 0), \quad (0, 1), \quad \text{and} \quad (1, 2).$$

PROBLEM 3. Slice the earth with a plane to get a circle. Use the intermediate value theorem to prove there are opposite points on this circle having the same temperature. (Describe the function to which you are applying the IVT and the assumptions that you are making in order for the hypotheses of the theorem to be satisfied.)

Solution. Parametrize the circle using the angle θ, and let $T(\theta)$ be the temperature at the point on the circle corresponding to θ. Then let $f(\theta) = T(\theta) - T(\theta + \pi)$ for $\theta \in [0, \pi]$. We are looking for a θ such than $f(\theta) = 0$. If $f(0) = 0$, we are done. Otherwise, note that
$$f(\pi) = T(\pi) - T(2\pi) = T(\pi) - T(0) = -f(0).$$
Since $f(0)$ and $f(\pi)$ have opposite signs, the IVT now applies. We are assuming that temperature varies continuously around the circle.

PROBLEM 4. Let $f\colon [a, b] \to \mathbb{R}$ be a continuous function, and suppose that $f(a) < 0$ and $f(b) > 0$. Describe an algorithm based on the intermediate value theorem that estimates a value $c \in (a, b)$ such that $f(c) = 0$. How quickly does this algorithm converge on a solution?

Solution. Use the *divide-and-conquer* algorithm. Let $c := \frac{a+b}{2}$ be the midpoint of the interval. If $f(c) = 0$, we are done. If $f(c) < 0$, we start again, this time using considering the function f restricted to the interval $[c, b]$. Since $f(c) < 0$ and $f(b) > 0$, the IVT guarantees f is zero somewhere in that interval. Otherwise, if $f(c) > 0$, we instead restrict f to $[a, c]$ and start again.

After n iterations, we have decreased the size of the interval in which we are searching by a factor of $1/2^n$, narrowing in on the point at which f vanishes. So the algorithm converges exponentially quickly.

In the lecture notes, we argued that

$$1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots = \left(1 - \frac{z^2}{\pi^2}\right)\left(1 - \frac{z^2}{4\pi^2}\right)\left(1 - \frac{z^2}{9\pi^2}\right)\cdots \tag{1}$$

and, by equating the coefficients of z^2 on each side, showed that

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

where $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$ is the Riemann zeta function. The point of the problems below is to show that $\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

PROBLEM 1. Imagine expanding the product on the right-hand side of (1). Each term in the expansion corresponds to making a choice in each factor between either 1 or $-\frac{z^2}{m^2\pi^2}$. Give a couple of examples of terms in the expansion that contribute to the coefficient of z^4. What does the general term contributing to the coefficient of z^4 look like?

Solution. A typical term contributing to z^4 has the form

$$\left(-\frac{z^2}{i^2\pi^2}\right)\left(-\frac{z^2}{j^2\pi^2}\right) = \frac{z^4}{i^2 j^2 \pi^4}$$

where $i \neq j$.

PROBLEM 2.

$$1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots = \left(1 - \frac{z^2}{\pi^2}\right)\left(1 - \frac{z^2}{4\pi^2}\right)\left(1 - \frac{z^2}{9\pi^2}\right)\cdots \tag{1}$$

Evaluate the coefficient of z^4 in the expansion of the right-hand side of (1) by looking at the left-hand side. (This should be easy.)

Solution. We get

$$\frac{1}{5!} = \frac{1}{120}.$$

PROBLEM 3.

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

Consider the following table

	$\frac{1}{1^2}$	$\frac{1}{2^2}$	$\frac{1}{3^2}$	$\frac{1}{4^2}$	$\cdots$
$\frac{1}{1^2}$	$\left(\frac{1}{1^2}\right)\left(\frac{1}{1^2}\right)$	$\left(\frac{1}{1^2}\right)\left(\frac{1}{2^2}\right)$	$\left(\frac{1}{1^2}\right)\left(\frac{3}{1^2}\right)$	$\left(\frac{1}{1^2}\right)\left(\frac{1}{4^2}\right)$	$\cdots$
$\frac{1}{2^2}$	$\left(\frac{1}{2^2}\right)\left(\frac{1}{1^2}\right)$	$\left(\frac{1}{2^2}\right)\left(\frac{1}{2^2}\right)$	$\left(\frac{1}{2^2}\right)\left(\frac{1}{3^2}\right)$	$\left(\frac{1}{2^2}\right)\left(\frac{1}{4^2}\right)$	$\cdots$
$\frac{1}{3^2}$	$\left(\frac{1}{3^2}\right)\left(\frac{1}{1^2}\right)$	$\left(\frac{1}{3^2}\right)\left(\frac{1}{2^2}\right)$	$\left(\frac{1}{3^2}\right)\left(\frac{1}{3^2}\right)$	$\left(\frac{1}{3^2}\right)\left(\frac{1}{4^2}\right)$	$\cdots$
$\frac{1}{4^2}$	$\left(\frac{1}{4^2}\right)\left(\frac{1}{1^2}\right)$	$\left(\frac{1}{4^2}\right)\left(\frac{1}{2^2}\right)$	$\left(\frac{1}{4^2}\right)\left(\frac{1}{3^2}\right)$	$\left(\frac{1}{4^2}\right)\left(\frac{1}{4^2}\right)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

(a) Why is the sum of all of the entries in the table is $\zeta(2)^2$.

(b) What is the sum of the terms on the diagonal in terms of the zeta function?

(c) What is the sum of the terms off of the diagonal? (Hint: see Problems 1 and 2 in order to find a numerical value.)

Solution.

(a) We have
$$\zeta(2)^2 = \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)$$
The terms in the expansion of the product have the form
$$\frac{1}{i^2} \cdot \frac{1}{j^2},$$
with $i, j \in \mathbb{Z}_{\geq 1}$. These are exactly the entries in the table.

(b) The sum of the diagonal terms is
$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

(c) By Problems 1 and 2
$$\sum_{\substack{i,j \in \mathbb{Z}_{\geq 1} \\ i \neq j}} \frac{1}{i^2 j^2 \pi^2} = \frac{1}{120}.$$
The sum of the off-diagonal terms contains that sum twice. Hence, the sum of the off-diagonal terms is
$$2 \sum_{\substack{i,j \in \mathbb{Z}_{\geq 1} \\ i \neq j}} \frac{1}{i^2 j^2} = 2 \cdot \frac{\pi^4}{120}. = \frac{\pi^4}{60}$$

Problem 4. Use Problem 3 to show that
$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Solution. By Problem 3 the sum of all entries in the table is $\zeta(2)^2$. On the other hand, we can break the sum into two parts: the off-diagonal entries, whose sum is $\pi^2/120$, and the diagonal entries, whose sum is $\zeta(4)$. Therefore,

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \zeta(2)^2 - \frac{\pi^4}{120} = \left(\frac{\pi^2}{6}\right)^2 - \frac{\pi^4}{60} = \left(\frac{1}{36} - \frac{1}{60}\right)\pi^4 = \left(\frac{10-6}{360}\right)\pi^4 = \frac{\pi^4}{90}.$$