Math 112 lecture for Wednesday, Week 13

(Supplemental reading: Sections 5.2 and 5.3 in Swanson.)

Two theorems from calculus

The extreme value theorem. We present proofs of two basic theorems whose proofs are usually skipped in a first course in calculus: the extreme value theorem and the intermediate value theorem.

The extreme value theorem is of fundamental importance in the theory of optimization since it states conditions under which the existence of a maximal and minimal value for a function is guaranteed.

Theorem (extreme value theorem, EVT). If $f\colon [a,b] \to \mathbb{R}$ is continuous, then f achieves is maximum and minimum on $[a,b]$, i.e., there exists $x_{\max}, x_{\min} \in [a,b]$ such that

$$f(x_{\min}) \leq f(x) \leq f(x_{\max})$$

for all $x \in [a,b]$.

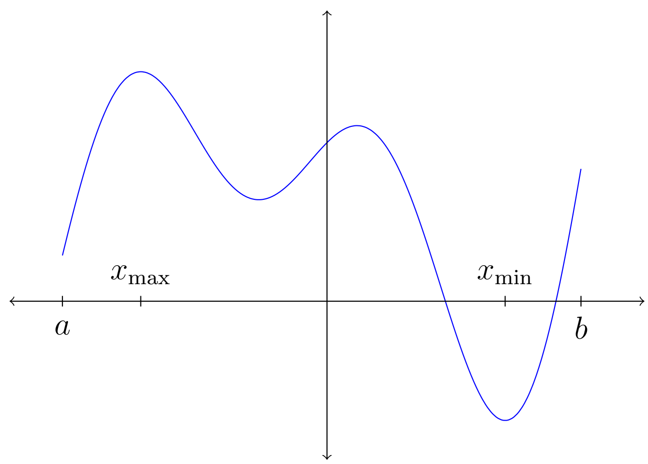


Figure 1: A continuous function on a closed interval $[a,b]$ with the points at which it achieves its (global) minimum and maximum labeled.

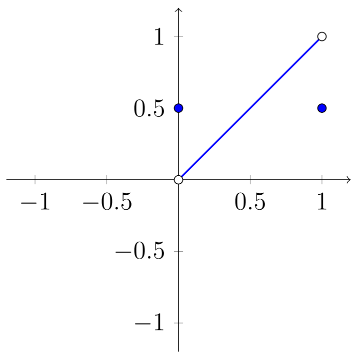
There are three important hypotheses in the EVT: (i) f is continuous, (ii) the interval is closed, and (iii) the interval is bounded.[1] The following three examples show that if you violate any one of these, the theorem no longer holds.

[1]Note that (ii) does not imply (iii). For instance, the intervals $[0,\infty)$ and $(-\infty,\infty)$ are both closed—their complements are open—but not bounded.

Example 1. Let $f\colon [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ \frac{1}{2} & \text{if } x = 0 \text{ or } 1. \end{cases}$$

Then f is defined on a closed and bounded interval but is not continuous. It has no maximum or minimum:

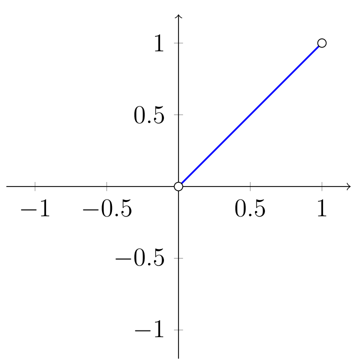


Graph of f.

Example 2. The identity function

$$\begin{aligned} g\colon (0,1) &\to (0,1) \subset \mathbb{R} \\ x &\mapsto x \end{aligned}$$

is continuous and defined on a bounded interval, but the interval is not closed. The function g has no minimum or maximum:



Graph of g.

The function $x \mapsto 1/x$ with domain $(0, 1)$ is a similar example.

Example 3. The identity function

$$\begin{aligned} h\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x \end{aligned}$$

is continuous on the closed but unbounded interval $(\infty, \infty) = \mathbb{R}$. It has no minimum or maximum.

To prove the extreme value theorem we need a couple of preliminary results. First, recall that if $X \subseteq \mathbb{R}$ and $g\colon X \to \mathbb{R}$ is a function, then g is *bounded* if there exists a constant B such that $|g(x)| \leq B$ for all $x \in [a, b]$. Also, g is *continuous* at $c \in [a, b]$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - c| < \delta$ implies $|g(x) - g(c)| < \varepsilon$.

Lemma. If $f\colon [a, b] \to \mathbb{R}$ is continuous, then f is locally bounded, i.e., given $c \in [a, b]$, there exists $\delta > 0$ such that f is bounded on $(c - \delta, c + \delta) \cap [a, b]$.[2]

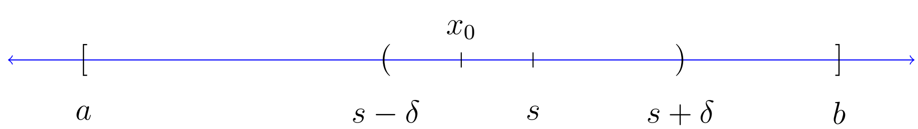
Proof. Apply the definition of continuity at c with $\varepsilon = 1$. We find a $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon = 1$. Therefore, $f(c) - 1 < f(x) < f(c) + 1$ for $x \in (c - \delta, c + \delta) \cap [a, b]$. (So we could take $B = \max\{|f(c) - 1|, |f(c) + 1|\}$.) $\square$

Proposition. If $f\colon [a, b] \to \mathbb{R}$ is continuous, then it is bounded.

Proof. This proof is a sort of continuous version of induction. Define

$$A = \{x \in [a, b] : f \text{ is bounded on } [a, x]\}.$$

To see that the supremum of A exists, note that A is nonempty since $a \in A$, and A is bounded above by b. Hence, by completeness, $s := \sup(A)$ exists. Our next goal is to show $s = b$. We will prove this by contradiction. Since b is an upper bound for A, and s is the least upper bound, we know that $s \leq b$. So suppose that $s < b$. By the Lemma, there exists $\delta > 0$ such that f is bounded on $(s - \delta, s + \delta)$ (by taking δ small enough, we may assume that $(s - \delta, s + \delta) \subset [a, b]$):



[2]Thus, f is bounded in a "neighborhood" about each point $c \in [a, b]$.

Since $s-\delta < s$, i.e., $s-\delta$ is less than the least upper bound for A, it follows that $s-\delta$ is not an upper bound for A. Therefore, there exists $x_0 \in A$ such that $s-\delta < x_0$. Since $x_0 \in A$, we have that f is bounded on $[a, x_0]$. But since f is also bounded on $(s-\delta, s+\delta)$, it then follows that it is bounded on

$$[a, x_0] \cup (s-\delta, s+\delta) = [a, s+\delta).$$

In particular, f is bounded on $[a, s+\delta/2]$. So $s+\delta/2 \in A$, contradicting the fact that $s = \sup(A)$. We conclude that, in fact, $s = b$.

Finally, we show that $b \in A$ by using an argument similar to the one we just used. We have established the fact that $b = \sup(A)$. By the Lemma, we know that there exists $\delta > 0$ such that f is bounded on $(b-\delta, b+\delta) \cap [a,b] = (b-\delta, b]$. Since b is the least upper bound for A and $b-\delta$ is less than b, it follows that $b-\delta$ is not an upper bound for A. So there exists $x_1 \in A$ such that $b-\delta < x_1$. Hence, f is bounded on

$$[a, x_1] \cup (b-\delta, b] = [a, b],$$

as required. □

We now know that a continuous function on a closed bounded interval is bounded. This is the key tool in proving the extreme value theorem.

Proof of the extreme value theorem. We just need to prove that $x_{\max}$ always exists. If we know that, then to prove that $x_{\min}$ exists, apply our result to the function $-f$ (which also continuous on $[a, b]$). The maximum we find for $-f$ will be a minimum for f.

Consider the set

$$\operatorname{im}(f) = f([a, b]) = \{f(x) : x \in [a, b]\}.$$

By our Proposition, f is bounded on $[a, b]$, which is equivalent to saying the set $\operatorname{im}(f)$ is a bounded set. Since $\operatorname{im}(f)$ is also nonempty, e.g., $f(a) \in \operatorname{im}(f)$, we can let $s := \sup(\operatorname{im}(f))$. Our proof is finished if we can show that $s \in \operatorname{im}(f)$ for in that case, there exists some $\tilde{x} \in [a, b]$ such that $f(\tilde{x}) = s$, and $f(\tilde{x}) \geq f(x)$ for all $x \in [a, b]$.

We prove $s \in \operatorname{im}(f)$ by contradiction. Suppose it is not, and define

$$\begin{aligned} g \colon [a, b] &\to \mathbb{R} \\ x &\mapsto \frac{1}{s - f(x)}. \end{aligned}$$

Since, by supposition, $s \notin \operatorname{im}(f)$, the function g is well-defined (we never divide by 0) and continuous. By the Proposition, g is therefore bounded. However, we will now show that g can't be bounded. Since $s = \sup(\mathrm{im}(f))$, there are points $s' \in \mathrm{im}(f)$ that are arbitrarily close to s, and writing $s' = f(x')$, we see $g(x') = \frac{1}{s-f(x')} = \frac{1}{s-s'}$ will be arbitrarily large. In detail, given any potential bound $B \in \mathbb{R}$, choose $\varepsilon > 0$ such that $\frac{1}{\varepsilon} > B$. Next, consider $s - \varepsilon$. Since it is less than the least upper bound s for $\mathrm{im}(f)$ and $s \notin \mathrm{im}(f)$, there exists $s' \in \mathrm{im}(f)$ such that $s - \varepsilon < s' < s$. Since $s' \in \mathrm{im}(f)$, there exists $x' \in [a,b]$ such that $f(x') = s'$. We then have

$$g(x') = \frac{1}{s - f(x')} = \frac{1}{s - s'} > \frac{1}{s - (s - \varepsilon)} = \frac{1}{\varepsilon} = B.$$

Hence, g has no upper bound B. $\square$

The intermediate value theorem. The intermediate value theorem is intuitively obvious, but its proof requires familiarity with the completeness property of the real numbers.

Theorem (intermediate value theorem, IVT). Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function such that $f(a) < 0$ and $f(b) > 0$. Then there exists $s \in (a,b)$ such that $f(s) = 0$.

Proof. Define the set

$$A := \{x \in [a,b] : f \text{ is negative at all points in } [a,x]\}.$$

We have $a \in A$ since $f(a) < 0$. Thus, $A \neq \emptyset$. Since $A \subseteq [a,b]$, the set A is bounded above by b. By completeness of $\mathbb{R}$ is follows that $s := \sup(A)$ exists. We are done if we can show that $f(s) = 0$.

We first argue that $s \notin \{a,b\}$. First, since $f(a) < 0$ and f is continuous, there is a $\delta_a > 0$ such that f is negative on the interval $[a, a+\delta_a)$. This shows $s \neq a$. Next, since $f(b) > 0$ and f is continuous, there exists a $\delta_b > 0$ such that f is positive on $(b - \delta_b, b]$. Hence, $s \neq b$. Therefore, $s \in (a,b)$.

Now note that if $f(c) > 0$ for any point $c \in [a,b]$, then c is an upper bound for A. If it were not, then there would an $x \in A$ such that $c < x$. But then, by definition of A, the function f is negative on $[a,x]$ and since $c \in [a,x]$, that would mean $f(c) < 0$. In particular, since s is the least upper bound for A, the function f cannot be positive at any point strictly less than s. Thus, we have shown that

$$[a,s) \subseteq A.$$

For sake of contradiction, suppose that $f(s) < 0$. By continuity of f, there exists $\delta > 0$ such that f is negative on $(s-\delta, s+\delta) \cap [a,b]$. Since $s \in (a,b)$, we can take δ small enough to that $(s-\delta, s+\delta) \cap [a,b] = (s-\delta, s+\delta)$. But then f is negative on

$$[a,s) \cup (s + \delta/2] = [a, s+\delta/2],$$

which shows that $s + \delta/2 \in A$, contradicting the fact that s is an upper bound for A.

Next, for the sake of contradiction, suppose that $f(s) > 0$. By continuity of f it follows that f is positive in a small interval $(s - \eta, s + \eta)$. Since $s \notin \{a, b\}$, we can assume $(s - \eta, s + \eta) \subset A$. Then, for instance, letting $c := s - \eta/2$, we see that $f(c) > 0$. As discussed above, that means that c is an upper bound for A. But that's impossible since $c < s$ and s is the least upper bound for A.

We conclude that $f(s) = 0$, as desired. $\square$