Math 112 lecture for Monday, Week 13

(Supplemental reading: Sections 6.5 and 9.7–9 in Swanson.)

Taylor series II

Convergence of Taylor series. Given a function $f(z)$ whose derivatives of all orders exist at a point a, it makes sense to compute the Taylor series of $f(z)$ centered at a:

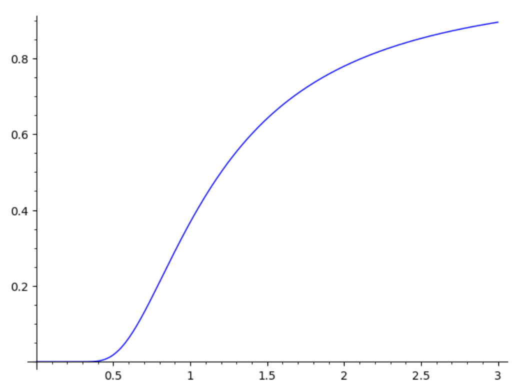
$$T(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n.$$

Then $T(z)$ is a power series in $z-a$, and so it has a radius of convergence R. The question arises: at points z inside the disk of convergence (centered at a), i.e., where $|z-a| < R$, is it true that $f(z) = T(z)$. Over the complex numbers, the answer is "yes", but over the real numbers things are more complicated:

Example. Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-1/x^2} & \text{if } x > 0. \end{cases}$$

Its graph looks like this:



It turns out that the derivatives of f of all orders exist at 0, and they are all 0. Hence, its Taylor polynomial is 0. However, $f(x) > 0$ for all $x > 0$. So f is not equal to its Taylor polynomial centered at 0 on any interval about 0.

In the last lecture, we saw that over the reals, the error term in approximating the function f with by its n-th Taylor polynomial is

$$|f(x) - T_n(x)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}\right| |x-a|^{n+1} \tag{1}$$

where c is a point between x and a. Suppose that $f^{(n+1)}$ is bounded in some interval containing x and a by a constant, independent of n. In other words, suppose there is a constant B such that for all n,

$$\left|f^{(n+1)}(c)\right| \leq B$$

for all c in some interval containing x and a. In that case, we have

$$0 \leq \lim_{n\to\infty} |f(x) - T_n(x)| = \lim_{n\to\infty} \left|\frac{f^{(n+1)}(c)}{(n+1)!}\right| |x-a|^{n+1} \leq \lim_{n\to\infty} \frac{B}{(n+1)!}|x-a|^{n+1} = 0.$$ [1]

It is easy to check that $x \mapsto |x|$ is a continuous function and, hence, commutes with limits. This means that

$$0 = \lim_{n\to\infty} |f(x) - T_n(x)| = \left|\lim_{n\to\infty} f(x) - T_n(x)\right| = |f(x) - T(x)|.$$

Therefore, in this case, the real function f will converge to its Taylor series.

Example. The real Taylor series for $\cos(x)$ centered at 0 is

$$T(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots .$$

An easy application of the ratio test shows that its radius of convergence is $R = \infty$. Then at any point c we have that $\cos^{(n)}(c)$ is either $\pm\cos(c)$ or $\pm\sin(c)$, which are all bounded in absolute value by $B = 1$. Therefore, $\cos(x) = T(x)$ for all $x \in \mathbb{R}$.

Example. For a more problematic example, consider the Taylor series for $f(x) = \ln(x)$ centered at $x = 1$. We have

$$f'(x) = \frac{1}{x},\ f''(x) = -\frac{1}{x^2},\ f^{(3)}(x) = \frac{2\cdot 1}{x^3},\ f^{(4)}(x) = -\frac{3\cdot 2\cdot 1}{x^4}, f^{(5)}(x) = \frac{4\cdot 3\cdot 2\cdot 1}{x^5},$$

and so on: $f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{x^n}$. In general, for $n > 0$, the n-th coefficient for the Taylor series centered at 1 is

$$\frac{f^{(n)}(1)}{n!} = (-1)^{n-1}\frac{(n-1)!}{n!} = \frac{(-1)^{n-1}}{n}.$$

Thus, the Taylor series for f centered at 1 is

$$T(x) := \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

[1] For any real or complex number α, we have $\lim_{n\to\infty} \frac{\alpha^n}{n!} = 0$. A roundabout way to see this is to note that $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ has radius of convergence $R = \infty$. Hence, for all z, the sequence of terms of the series converges to 0.

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(n-1)!}{n!} (x-1)^n$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} (x-1)^n$$

$$= (x-1) - \frac{(x-1)}{2} + \frac{(x-1)^2}{3} - \frac{(x-1)^4}{4} + \cdots .$$

We can use the power series ratio test find the radius of convergence:

$$\lim_{n\to\infty} \left| \frac{(-1)^{n-1}}{n} \right| \Big/ \left| \frac{(-1)^n}{n+1} \right| = \lim_{n\to\infty} \frac{n+1}{n} = 1.$$

Hence, the series converges (absolutely) for $|x-1| < 1$ and diverges for $|x-1| > 1$. Thus, in the open interval of radius 1 centered at 1, i.e., in $(0, 2)$, the series converges absolutely. What happens on the boundary? At $x = 2$, we get the alternating harmonic series

$$T(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots,$$

which converges by the alternating series test. At $x = 0$, we have

$$T(0) = -1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots ,$$

which diverges since its partial sums differ from those of the harmonic series by a constant.

Does the Taylor series for $\ln(x)$ converge to $\ln(x)$ on $(0, 2)$? It turns out that it does, but the argument depends on a different formulation for the error term when approximating a function by a Taylor polynomial. There is a further interesting question: what happens when $x = 2$, on the boundary of the interval of convergence? We have seen above that $T(1)$ is the alternating harmonic series. Is it true that $\ln(2) = T(2)$ The answer is, again, yes. The details appear in the appendix at the end of this lecture for those who are interested.

The complex exponential function

The usual exponential, cosine, and sine functions generalize to functions on the whole complex plane. Moreover, the generalization leads to a hidden relation among all three functions (Euler's formula).

Definition. For $z \in \mathbb{C}$ define the power series

$$E(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

$$C(z) := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

$$S(z) := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots .$$

Exercise. Here are a few straightforward exercises concerning these functions:

(a) Use the ratio test to show that $E(z)$, $C(z)$, and $S(z)$ converge for all $z \in \mathbb{C}$.

(b) Show that $C(-z) = C(z)$ and $S(-z) = -S(z)$.

(c) Show that

$$E'(z) = E(z), \quad C'(z) = -S(z), \quad \text{and} \quad S'(z) = C(z).$$

(d) Show that $C^2(z) + S^2(z) = 1$. (Hint: use derivatives.)

(e) For $z \in \mathbb{R}$, a real number, show that these functions are the Taylor series for e^z, $\cos(z)$, and $\sin(z)$.

Proposition. For $x \in \mathbb{R}$,

$$E(x) = e^x, \quad C(x) = \cos(x), \quad \text{and} \quad S(x) = \sin(x).$$

Proof. It suffices to show that e^x, $\cos(x)$, and $\sin(x)$ equal their Taylor series. That follows from the error term for the approximation of a function by a Taylor polynomial which we have discussed previously. For instance, let $f(x) = e^x$, and let $T_n(x)$ be the n-th order Taylor polynomial for e^x. Given $x \in \mathbb{R}$, there exists c between x and 0 such that

$$e^x - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} = \frac{e^c x^{n+1}}{(n+1)!}.$$

Therefore,

$$e^x - T(x) = \lim_{n\to\infty} (e^x - T_n(x)) = e^c \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0.^2$$

[2] See the previous footnote for an proof that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.

We gave the argument for $\cos(x)$ in an earlier example, and the example for $\sin(x)$ is similar. $\square$

Thus, for instance, we know that $E(\sqrt{2}) = e^{\sqrt{2}}$. What can we say about something like e^{2+3i}? What about $\cos(2+3i)$? It turns out that up to this point, we have no definition of what it means take the exponential, cosine, or sine of a complex number. So we are free to decide what these mean, as we do in the following definition:

Definition. For $z \in \mathbb{C}$ let

$$\exp(z) = E(z), \quad \cos(z) = C(z), \quad \sin(z) = S(z).$$

Proposition (Euler's formula). For all $z \in \mathbb{C}$, we have

$$e^{iz} = \cos(z) + i\sin(z).$$

Proof. We have

$$\begin{aligned}
\exp(iz) &= 1 + (iz) + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \frac{(iz)^5}{5!} + \frac{(iz)^6}{6!} + \dots \\
&= 1 + (iz) + \frac{i^2z^2}{2!} + \frac{i^3z^3}{3!} + \frac{i^4z^4}{4!} + \frac{i^5z^5}{5!} + \frac{i^6z^6}{6!} + \dots \\
&= 1 + iz - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} - \frac{z^6}{6!} + \dots \\
&= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots\right) \\
&= \cos(z) + i\sin(z).
\end{aligned}$$

$\square$

Thus, for instance, we may parametrize the unit circle centered at the origin by

$$\begin{aligned} \mathbb{R} &\to \mathbb{C} \\ t &\mapsto e^{it}. \end{aligned}$$

Corollary. For $z \in \mathbb{C}$

$$\cos(z) = \frac{\exp(iz) + \exp(-iz)}{2} \quad \text{and} \quad \sin(z) = \frac{\exp(iz) - \exp(-iz)}{2i}.$$

Proof. Exercise. □

One thing that is not immediately clear from our definition of e^z is that it obeys the usual exponent laws for all complex numbers! We show that now.

Proposition. Let $w, z \in \mathbb{C}$. Then

(a) $\exp(-z) = \dfrac{1}{\exp(z)}$.

(b) $\exp(w+z) = \exp(w)\exp(z)$

Proof. For part (a), fix $w \in \mathbb{C}$ and define the function $f(z) = \exp(w+z)\exp(-z)$. Apply the product and chain rules for differentiation with respect to z to find

$$\begin{aligned} f'(z) &= \exp'(w+z)\exp(-z) + \exp(w+z)\exp'(-z) \\ &= \exp(w+z)\exp(-z) - \exp(w+z)\exp(-z) \\ &= 0. \end{aligned}$$

It follows that $f(z) = c$ for some constant c. To find the constant, we just need to evaluate f at any point, say $z = 0$. We find $c = f(0) = \exp(w+0)\exp(0) = \exp(w)$. Hence, $f(z) = f(w)$ for all $z \in \mathbb{C}$:

$$\exp(z+w)\exp(-z) = \exp(w). \tag{2}$$

Equation (2) holds for all $w, z \in \mathbb{C}$. In particular, setting $w = 0$, we find

$$\exp(z)\exp(-z) = \exp(0) = 1.$$

This proves part (a). Part (b) then immediately follows from (a) and (2). □

Remark. Given $z = x + iy \in \mathbb{C}$ where $x, y \in \mathbb{R}$, we have

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y)).$$

Note that on the right, we have the polar form for the complex number e^z. Hence,

$$|e^z| = e^x \quad \text{and} \quad \arg(e^z) = y.$$

In general, exponential notation arises naturally when writing a complex number in polar form:

$$w = r(\cos(\theta) + \sin(\theta)) = re^{i\theta}.$$

The geometry of multiplication of complex numbers—that lengths multiply and angles add—takes the form

$$(re^{i\theta})(se^{i\psi}) = rse^{i\theta + i\psi} = rse^{i(\theta+\psi)}.$$

Appendix

We would like to show that $\ln(x)$ converges to its Taylor series centered at 1 on the interval $(0, 2]$. Earlier, we calculated the derivatives

$$\ln^{(n+1)}(c) = (-1)^n \frac{n!}{c^{n+1}}$$

and found the Taylor series to be

$$T(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n} = (x-1) - \frac{(x-1)}{2} + \frac{(x-1)^2}{3} - \frac{(x-1)^4}{4} + \cdots,$$

which converges on $(0, 2]$.

To show $\ln(x) = T(x)$ on this interval takes several steps. First note that the case $x = 1$ is trivial: $\ln(1) = 0 = T(1)$. The case of the endpoint $x = 2$ will be handled at the end. We first consider the two cases $0 < x < 1$ and $1 < x < 2$ separately. For the latter, we will need to use an alternative form of the error term.

Let $T_n(x)$ denote the n-th Taylor polynomial for $\ln(x)$. For the case where $0 < x < 1$, we use Equation (1) to get

$$|\ln(x) - T_n(x)| = \frac{1}{(n+1)!} \frac{n!}{c^{n+1}} |x-1|^{n+1} = \frac{1}{n+1} \cdot \left(\frac{1-x}{c}\right)^{n+1}.$$

where $0 < x < c < 1$. In this case, $1 - x < c$, and therefore

$$\frac{1}{n+1} \cdot \left(\frac{1-x}{c}\right)^{n+1} < \frac{1}{n+1} \longrightarrow 0,$$

as $n \to \infty$. Hence, we see that $\ln(x) = T(x)$ on $(0, 1)$.

For the next case, we need a different form for the remainder.

Theorem. Let $A \subseteq \mathbb{R}$ and let $f\colon A \to \mathbb{R}$. Suppose that A contains an open interval containing the closed interval $[u, v]$. Also suppose that all the derivatives up to order n exist and are continuous on $[u, v]$ and the $(n+1)$-th derivative exists on (u, v). Let $a \in A$, and let $T_{n,a}$ be the Taylor polynomial of order n for f centered at a. Then for all $x \in [u, v]$, there exists a number c strictly between x and a such that

$$f(x) = T_{n,a}(x) + \frac{1}{n!} f^{(n+1)}(c)(x-a)(x-c)^n. \tag{3}$$

Proof. See Swanson, Theorem 6.5.5. □

We apply this theorem to $\ln(x)$ in the case $1 < x < 2$ to see

$$|\ln(x) - T_n(x)| = \frac{1}{n!}\frac{n!}{c^{n+1}}(x-1)(x-c)^n = \frac{(x-1)(x-c)^n}{c^{n+1}}$$

for some c such that $1 < c < x < 2$. Since $1 < c$ and $0 < x - c < x - 1 < 1$, it follows that

$$0 \leq \frac{(x-1)(x-c)^n}{c^{n+1}} \leq (x-1)^{n+1} \longrightarrow 0$$

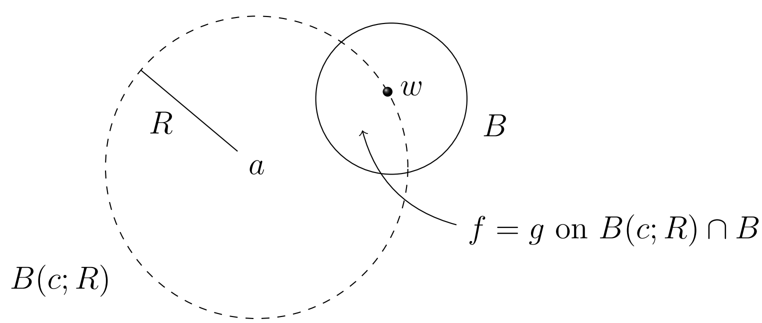
as $n \to \infty$.

Having show that $\ln(x)$ equals its Taylor series on $(0, 2)$, we now consider the point $x = 2$. For that, we apply the following

Theorem. Let $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ be a real or complex power series with radius of convergence $R > 0$. Let c be a point on the boundary of the ball of convergence, i.e., such that $|c| = R$. Let B be any open ball centered at c, and let g be a continuous function with domain B such that $f(z) = g(z)$ on $B(c; R) \cap B$. Then, $f(c) = g(c)$.

Proof. See Swanson, Theorem 9.6.2. $\square$

Picture for the case of complex series:



Now apply this theorem with $g(x) = \ln(x)$ and $f(x) = T(x)$ to conclude

$$\ln(2) = T(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$