Math 112 lecture for Friday, Week 13

Three theorems by Euler

Theorem 1. The number e is irrational.

Proof. Euler discovered that $e = \sum_{n=0}^{\infty} \frac{1}{n!}$, and used that to prove that e is irrational. To start, we can at least see that e is an integer since

$$2 = 1 + \frac{1}{1!} < e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots < 1 + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right) = 3.$$

Now, for the sake of contradiction, suppose that e is rational, and write $e = \frac{p}{q}$ with $p, q \in \mathbb{Z}_{\geq 1}$. Since e is not an integer, $q > 1$. We have

$$\frac{p}{q} = e = \sum_{n=0}^{\infty} \frac{1}{n!} \quad \Longrightarrow \quad e - \sum_{n=0}^{q} \frac{1}{n!} = \sum_{n=q+1}^{\infty} \frac{1}{n!}$$

$$\Longrightarrow \quad q!\left(e - \sum_{n=0}^{q} \frac{1}{n!}\right) = \sum_{n=q+1}^{\infty} \frac{q!}{n!} \tag{1}$$

The expression on the left-hand side of (1),

$$q!\left(e - \sum_{n=0}^{q} \frac{1}{n!}\right),$$

is an integer since $e = \frac{p}{q}$ and $\frac{q!}{n!} \in \mathbb{Z}$ when $q \geq n$. We conclude that the right-hand side of (1) is also an integer:

$$\sum_{n=q+1}^{\infty} \frac{1}{n!} \in \mathbb{Z}.$$

However,

$$\sum_{n=q+1}^{\infty} \frac{q!}{n!} = \frac{q!}{(q+1)!} + \frac{q!}{(q+2)!} + \frac{q!}{(q+3)!} + \cdots$$

$$= \frac{1}{q+1} + \frac{1}{(q+2)(q+1)} + \frac{1}{(q+3)(q+2)(q+1)} + \cdots$$

$$< \frac{1}{q+1} + \frac{1}{(q+1)^2} + \frac{1}{(q+1)^3} + \cdots$$

$$= \frac{1}{q+1}\sum_{n=0}^{\infty}\left(\frac{1}{q+1}\right)^n$$

$$= \frac{1}{q+1}\cdot\frac{1}{1-\frac{1}{(q+1)}} \qquad \text{(geometric series formula)}$$

$$= \frac{1}{q} < 1.$$

Contradicting the fact that this sum is an integer. ☐

Theorem 2. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

Proof. We first recall some basic algebra. If $P(z)$ is a polynomial of degree n with real or complex coefficients, then it will have n (not necessarily distinct) roots, $r_1, \ldots, r_n$.[1] We can then write

$$P(z) = k(z - r_1)\cdots(z - r_n)$$

for some constant k. If no $r_i = 0$, we can rewrite this expression as

$$P(z) = (-1)^n k r_1 \cdots r_n \left(1 - \frac{z}{r_1}\right)\cdots\left(1 - \frac{z}{r_n}\right).$$

Now further assume that $P(0) = 1$. We then have

$$1 = P(0) = (-1)^n k r_1 \cdots r_n,$$

and, thus,

$$P(z) = \left(1 - \frac{z}{r_1}\right)\cdots\left(1 - \frac{z}{r_n}\right). \tag{2}$$

Consider $\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$, and divide by z to define

$$P(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \tag{3}$$

so that $P(z) = \frac{\sin(z)}{z}$ for $z \neq 0$, and at $z = 0$, we have.

$$P(0) = 1 = \lim_{z\to 0}\frac{\sin(z)}{z}.$$

[1] By *roots* we mean $P(r_k) = 0$ for each k.

Since $\sin(k\pi)$ is zero for all integers k, it follows that $P(k\pi) = 0$ for all nonzero integers $k = \pm 1, \pm 2, \ldots$ Therefore, skipping several technical details, we emulate (2) and write

$$\begin{aligned} P(z) &= \left(1 - \frac{z}{\pi}\right)\left(1 + \frac{z}{\pi}\right)\left(1 - \frac{z}{2\pi}\right)\left(1 + \frac{z}{2\pi}\right)\left(1 - \frac{z}{3\pi}\right)\left(1 + \frac{z}{3\pi}\right)\cdots \\ &= \left(1 - \frac{z^2}{\pi^2}\right)\left(1 - \frac{z^2}{4\pi^2}\right)\left(1 - \frac{z^2}{9\pi^2}\right)\cdots \end{aligned}$$

Going back to the definition (3) of P, we have shown that

$$1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots = \left(1 - \frac{z^2}{\pi^2}\right)\left(1 - \frac{z^2}{4\pi^2}\right)\left(1 - \frac{z^2}{9\pi^2}\right)\cdots$$

Comparing the coefficient of z^2 on both sides, we find

$$-\frac{1}{6} = -\sum_{n=1}^{\infty} \frac{1}{n^2\pi},$$

and the result follows. $\square$

Exercise. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

What about $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$? This number was shown to be irrational in 1978. It is an open question whether there is a rational r such that the sum is equal to $r\pi^3$.

Theorem 3. There are infinitely many prime numbers.

Proof. Let $s \in \mathbb{C}$. Then

$$\begin{aligned} \prod_{p \text{ prime}}^{\infty} \left(\frac{1}{1 - p^{-s}}\right) &= \prod_{p \text{ prime}}^{\infty} \left(1 + p^{-s} + p^{-2s} + p^{-3s} + \cdots\right) \\ &= \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \cdots\right)\left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \cdots\right)\cdots \end{aligned}$$

Patiently multiplying out the final expression, and using the fact the each positive integer can be uniquely factored into primes, you will find that

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}}^{\infty} \left(\frac{1}{1 - p^{-s}} \right)$$

If were only finitely many primes, the product on the right would be finite, and hence we could evaluate it as a complex number when $s = 1$. However,

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges. $\square$

Note: The function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{z^s}$ is known as the *Riemann zeta function*. Our previous result states that

$$\zeta(2) = \frac{\pi^2}{6}.$$