Math 112 lecture for Wednesday, Week 12

Power series II

(Supplemental reading: Sections 9.3 and 9.4 in Swanson.)

Our goal is to prove the following:

Theorem (Existence of radius of convergence.) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a real or complex power series. Then one of the following holds:

(a) $f(z)$ converges only when $z = 0$.

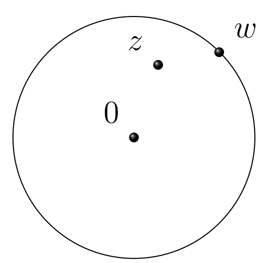
(b) $f(z)$ converges for all $z \in \mathbb{C}$.

(c) There exists a positive real number R such that $f(z)$ converges absolutely for $|z| < R$ and diverges for $|z| > R$.

Definition. The R in the above theorem is the *radius of convergence* of $f(z)$. We define $R = 0$ if $f(z)$ converges only when $z = 0$, and we take $R := \infty$ if $f(z)$ converges for all $z \in \mathbb{C}$.

We will need the following lemma:

Lemma. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a complex power series and suppose that $f(w)$ converges for some $w \in \mathbb{C} \setminus \{0\}$. Then $f(z)$ converges absolutely for all $z \in \mathbb{C}$ such that $|z| < |w|$:



Proof. Recall that if a series converges then the limit of its sequence of terms must go to zero and that convergent sequences are bounded. Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty} a_n w^n \text{ convergent} \quad &\Rightarrow \quad \lim_{n\to\infty} a_n w^n = 0 \\ &\Rightarrow \quad \{a_n w^n\} \text{ bounded}\end{aligned}$$

$$\Rightarrow \quad \exists M \in \mathbb{R}_{\geq 0} \text{ such that } |a_n w^n| \leq M \text{ for all } n.$$

Now suppose that $|z| < |w|$. We will use the comparison test to show that $\sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent. First note that

$$0 \leq |a_n z^n| = \left| a_n w^n \left(\frac{z}{w}\right)^n \right| = |a_n w^n| \left|\frac{z}{w}\right| \leq M \left|\frac{z}{w}\right|^n. \tag{1}$$

Define

$$r := \left|\frac{z}{w}\right|.$$

Then $|r| < 1$, so the following geometric series converges:

$$\sum_{n=0}^{\infty} M r^n = M \frac{1}{1-r}.$$

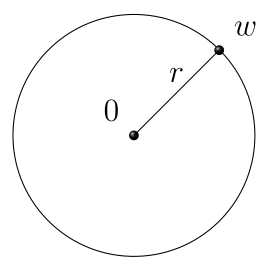
Hence, by (1) and the comparison theorem, $\sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent. □

Corollary. If $\sum_{n=0}^{\infty} a_n w^n$ diverges and $|z| > |w|$, then $\sum_{n=0}^{\infty} a_n z^n$ diverges.

Proof. If $\sum_{n=0}^{\infty} a_n z^n$ converged, then by the lemma, with the letters z and w swapped, $\sum_{n=0}^{\infty} a_n w^n$ would converge, too. □

Proof of the main theorem. Suppose that (a) and (b) do not hold. Then there exist various points $w \neq 0$ at which the series converges. We first collect the lengths of these w in a set:

$$S := \{r \in \mathbb{R} : \exists w \in \mathbb{C} \text{ s.t. } |w| = r \text{ and } f(w) = \textstyle\sum_{n=0}^{\infty} a_n w^n \text{ converges}\} :$$



Then S is a set of real numbers, and we'd like to show that S has a supremum. First, since $f(0)$ converges, $|0| = 0 \in S$. So S is nonempty. Next, we claim that S is bounded above. To see this, note that since (b) does not hold, there exists some $\tilde{w} \in \mathbb{C}$ such that $f(w)$ diverges. By the lemma, it follows that $f(z)$ diverges for all z such that $|z| > |\tilde{w}|$. Thus, S is bounded above by $|\tilde{w}|$. Therefore, by completeness of the real numbers, $\sup(S)$ exists. Define $R := \sup(S)$. Our next goal is to show that R is the radius of convergence for f.

On the one hand, if $|z| < R$, then $|z|$ is strictly less than the least upper bound for S. Hence, $|z|$ is not an upper bound for S. This means there exists $r \in S$ such that $|z| < r$. By definition of S, we have $r = |w|$ for some $w \in \mathbb{C}$ such that $f(w)$ converges. The lemma then implies that $\sum_{n=0}^{\infty} |a_n z^n|$ converges, i.e., $f(z)$ converges absolutely.

On the other hand, if $|z| > R$, then $|z|$ is greater than the least upper bound for S, which means that $|z| \notin S$, which in turn means that $f(z)$ does not converge. $\square$

Power series have a remarkable property with respect to differentiation. We state the result but do not include a proof:

Theorem (Differentiation of power series). Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R. Then f is differentiable for all z in the open ball of radius R centered at the origin

$$B(0; R) := \{w \in \mathbb{C} : |w| < R\},$$

and

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}.$$

Further, the radius of convergence for $f'(z)$ is R, the same as for f.

If f' has radius of convergence R, just like f, then we can apply the Theorem to f', to conclude that f'' also has radius of convergence R. We then apply the Theorem to f'' to conclude f''' has radius of convergence R, and so on.

Corollary. Inside its radius of convergence, a power series is infinitely differentiable.

Remark. Differentiability of a function is a measure of its *smoothness*. For instance, the graph of a differentiable function will not be pointy (as is, for example, that of the function $x \mapsto |x|$). If the second derivative exists, then the function is even more smooth. Thinking of the derivative as speed, this would mean that the speed does not change abruptly. If the third derivative exists, then the acceleration does not change abruptly, etc. By this measure, the smoothest functions are those that are infinitely differentiable.

Example. Most real functions, even differentiable ones, are not infinitely differentiable. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^{\frac{4}{3}}$. It is differentiable with derivative $f'(x) = \frac{4}{3} x^{\frac{1}{3}}$. However, $f''(0)$ does not exist. One conclusion we can draw from this is that f cannot be written as a power series with some positive radius of convergence.

Example. Consider the power series

$$f(z) = \sum_{n=0}^{\infty} z^n.$$

By our knowledge of geometric series (or by a quick calculation using the ratio test), we know that f has radius of convergence $R = 1$. So by the above theorem,

$$f'(z) = \sum_{n=0}^{\infty} nz^{n-1}$$

has radius of convergence 1. On the other hand, since f is a geometric series,

$$f(z) = \frac{1}{1-z}$$

for $|z| < 1$. Taking the derivative of both sides of this equation, we get

$$\sum_{n=0}^{\infty} nz^{n-1} = \frac{1}{(1-z)^2}$$

for $|z| < 1$. For instance, letting $z = 1/2$, we get

$$1 + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \cdots = \frac{1}{(1-1/2)^2} = 4.$$

So far, we have been considering power series "centered at 0". The balls in which they converge are centered at the origin. To provide more versatility, we make the following

Definition. Let $\{a_n\}$ be a sequence of complex numbers, and let $c \in \mathbb{C}$. The series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n$$

is a (complex) *power series centered at* c.

Remark. All of our previous results now apply to this more general context. A power series centered at c will have a radius of convergence R, and the series will be infinitely differentiable in the ball of radius R centered at c. To compute R, one may use the power series ratio test without modification, or one may use the ordinary ratio test, substituting $|z-c|$ for $|z|$.

Example. Consider the power series (centered at 3):

$$f(z) = \sum_{n=0}^{\infty} n^2 (z-3)^n.$$

Its radius of convergence may be computed using the power series ratio test:

$$R = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1.$$

Alternatively, we can use the ordinary ratio test:

$$\lim_{n\to\infty} \frac{(n+1)^2 |z-3|^{n+1}}{n^2 |z-3|^n} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} |z-3| = |z-3|.$$

We have $|z-3| < 1$ if and only if z is in the ball of radius 1 about the point 3.