Math 112 lecture for Monday, Week 12

Power series I

(Supplemental reading: Section 9.3 in Swanson.)

Definition. Let $\{a_n\}$ be a sequence of complex numbers. The series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is a (complex) *power series* with n-th coefficient a_n.

Remarks.

(a) A power series may be thought of as a family of ordinary series of the type we've just studied. We get one series for each point $z \in \mathbb{C}$. For instance, consider the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$. Letting $z = 1$ gives the series $\sum_{n=0}^{\infty} \frac{1}{n!}$, and letting $z = 2 + 3i$ gives $\sum_{n=0}^{\infty} \frac{(2+3i)^n}{n!}$.

(b) The *n-th term* of the series is $a_n z^n$, and the *n-th coefficient* is a_n.

(c) Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges for all $D \subseteq \mathbb{C}$. Then f defines a function $f\colon D \to \mathbb{C}$.

Theorem (Main theorem for power series.) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a complex power series. Then one of the following holds:

(a) $f(z)$ converges only when $z = 0$.

(b) $f(z)$ converges for all $z \in \mathbb{C}$.

(c) There exists a real number $R > 0$ such that $f(z)$ converges absolutely when $|z| < R$ and diverges for $|z| > R$.

Definition. The number R defined above is called the *radius of convergence* for the series. We say $R = 0$ in case (a) and $R = \infty$ in case (b) of the theorem.

We will prove this theorem next time. To find the radius of convergence, one usually uses the ratio test.

Example. Find the radius of convergence for $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

Solution. The ratio tests applies *after taking absolute values* of the terms:

$$\left|\frac{z^{n+1}}{(n+1)!}\right| \Big/ \left|\frac{z^n}{n!}\right| = \frac{n!}{(n+1)!} \cdot \frac{|z|^{n+1}}{|z|^n} = \frac{|z|}{n+1} \to 0,$$

as $n \to \infty$. So the ratio test says that the series converges absolutely for all $z \in \mathbb{C}$. Therefore, the radius of convergence is ∞.

Example. Find the radius of convergence of $f(z) = \sum_{n=0}^{\infty} \frac{1}{5^n} z^n$.

Solution. One could use the geometric series test here, but we'll use the ratio test again:

$$\left|\frac{1}{5^{n+1}} z^{n+1}\right| \Big/ \left|\frac{1}{5^n} z^n\right| = \frac{5^n}{5^{n+1}} \frac{|z|^{n+1}}{|z|^n} = \frac{|z|}{5} \to \frac{|z|}{5},$$

as $n \to \infty$. The ratio test says the series converges if

$$\frac{|z|}{5} < 1$$

and diverges if $\frac{|z|}{5} > 1$. We have

$$\frac{|z|}{5} < 1 \quad \Rightarrow \quad |z| < 5.$$

So the series converges absolutely for $|z| < 5$ and diverges for $|z| > 5$, i.e., the radius of convergence is 5. (On the boundary, where $|z| = 5$, we know the series diverges by the geometric series test: $\sum_{n=0}^{\infty} \left(\frac{z}{5}\right)^n$ diverges if $|z/5| \geq 1$.)

Example. Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(3n)!}{n!(2n)!} z^n$.

Solution. Apply the ratio test:

$$\begin{aligned}\left(\frac{(3(n+1))!}{(n+1)!(2(n+1))!}\right) |z|^{n+1} \Big/ \left(\frac{(3n)!}{n!(2n)!}\right) |z|^n &= \frac{(3(n+1))!}{(3n)!} \frac{n!}{(n+1)!} \frac{(2n)!}{(2(n+1))!} |z| \\ &= \frac{(3n+3)(3n+2)(3n+1)}{(n+1)(2n+2)(2n+1)} |z| \\ &\longrightarrow \frac{27}{4} |z|,\end{aligned}$$

as $n \to \infty$. By the ratio test, we get convergence if

$$\frac{27}{4}|z| < 1,$$

or in other words,

$$|z| < \frac{27}{4}.$$

So the radius of convergence is $\frac{27}{4}$.

A version of the ratio test for power series. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Apply the ratio test for series (remembering to take absolute values since the ratio test requires positive terms):

$$\frac{|a_{n+1}||z|^{n+1}}{|a_n||z|^n} = \frac{|a_{n+1}|}{|a_n|}|z| \to r|z|,$$

where

$$r = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \geq 0$$

which we are supposing exists. The ratio test then says we get convergence if

$$r|z| < 1.$$

If $r = 0$, the power series convergences for all z, and the radius of convergence is ∞. Otherwise, the power series converges for z satisfying

$$|z| < \frac{1}{r},$$

and the radius of convergence is $1/r$. However, note that

$$\frac{1}{r} = \frac{1}{\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}} = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}.$$

If $r = 0$, we'll have $\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \infty$. So we get the following result:

Proposition (Ratio test for power series). Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a complex power series with (eventually) nonzero coefficients, and suppose that

$$\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = R \in \mathbb{R} \cup \{\infty\}.$$

Then R is the radius of convergence of f.

WARNING: Note that the ratio test for an ordinary series of positive terms $\sum_{n=0}^{\infty} a_n$ requires checking whether

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1.$$

Compare this with the limit in the proposition. There are two differences: (i) We take absolute values since we are not assuming the terms in the power series are positive, and (ii) The order in which a_n and a_{n+1} appear in the numerator and denominator switch.

Example. Repeating the previous example using the power series ratio test rather than the ordinary ratio test we would do the following calculation:

$$\begin{aligned} \left(\frac{(3n)!}{n!(2n)!}\right) \Big/ \left(\frac{(3(n+1))!}{(n+1)!(2(n+1))!}\right) &= \frac{(3n)!}{(3(n+1))!} \frac{(n+1)!}{n!} \frac{(2(n+1))!}{(2n)!} \\ &= \frac{(n+1)(2n+2)(2n+1)}{(3n+3)(3n+2)(3n+1)} \\ &\longrightarrow \frac{4}{27}, \end{aligned}$$

as $n \to \infty$. Using the power series ratio test is quicker than using the usual ratio test since we leave out factors of $|z|$ and we don't have to solve for $|z|$ at the end.

Example. Here is an example from the Math 112 notes by Ray Mayer (Math 112 Notes). Consider the power series

$$f(z) = \sum_{n=0}^{\infty} \frac{z^{n^2}}{n^2}.$$

What is its radius of convergence? The ratio test and power series ratio test do not directly apply to $f(z)$ since its k-th coefficient is 0 unless $k = n^2$ for some integer n (and, hence, we cannot divide by it). In this case, define $a_n := \frac{z^{n^2}}{n^2}$, and then

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{z^{n^2}}{n^2}.$$

We can then apply the ordinary ratio test to $\sum_{n=0}^{\infty} a_n$, which is a series of nonzero terms:

$$\frac{|a_{n+1}|}{|a_n|} = \left|\frac{z^{(n+1)^2}}{(n+1)^2}\right| \Big/ \left|\frac{z^{n^2}}{n^2}\right| = \frac{n^2}{(n+1)^2} \frac{|z|^{n^2+2n+1}}{|z|^{n^2}} = \frac{n^2}{(n+1)^2} |z|^{2n+1}.$$

We have

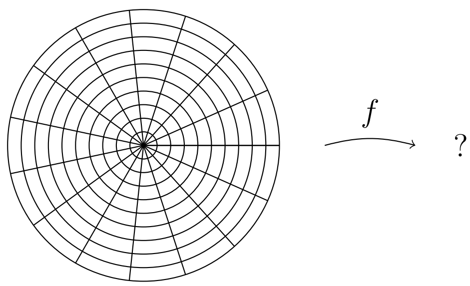
$$\lim_{n\to\infty} \frac{n^2}{(n+1)^2}|z|^{2n+1} = \begin{cases} 0 & \text{if } |z| < 1 \\ \infty & \text{if } |z| > 1. \end{cases}$$

It follows that the radius of convergence for the original series is $R = 1$. When $|z| = 1$, the series converges absolutely since $\sum_{n=0}^{\infty} \frac{1}{n^2}$ converges by the p-series test.

Let $D = \{z \in \mathbb{C} : |z| \leq 1\}$ be the closed unit disk, and consider the function

$$\begin{aligned} f\colon D &\to \mathbb{C} \\ z &\mapsto \sum_{n=0}^{\infty} \frac{z^{n^2}}{n^2}. \end{aligned}$$

To picture f draw the following picture in the plane, centered at the origin, and look at its image after applying f:



The image is pictured below:

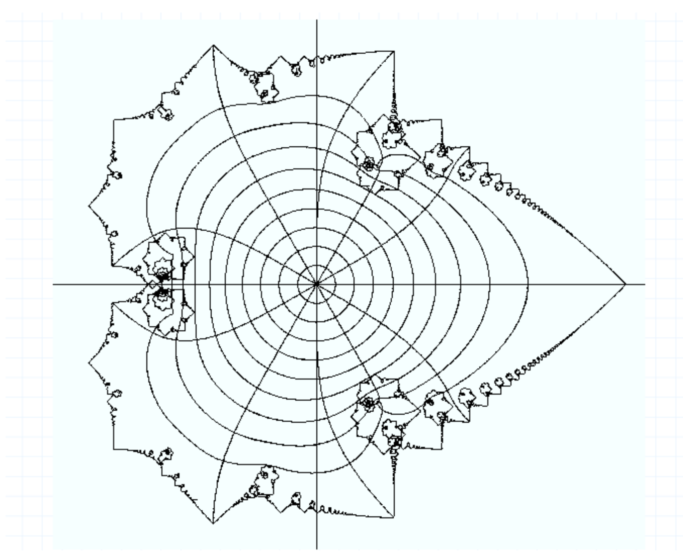


Image of concentric circles and radial lines under $f(z) = \sum_{n=0}^{\infty} \frac{z^{n^2}}{n^2}$.

The fractal-like boundary is the image of the boundary of the disc. We see that when z is small, the function looks almost like the identity function—sending circles in the domain to near-circles in the codomain. As z approaches the boundary of the disc, f gets more and more "confused".