Math 112 lecture for Friday, Week 12

(Supplemental reading: Sections 6.5 and 9.7 in Swanson.)

Taylor series I

The purpose of Taylor series is to approximate functions by polynomials.

Suppose that $f\colon D \to \mathbb{C}$ for some open set $D \subseteq \mathbb{C}$, and let $a \in D$. Suppose that the k-th derivative $f^{(k)}(a)$ exists for all $k \geq 0$.

Recall that by definition,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Therefore, if h is small, we have

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}.$$

Solving for $f(a+h)$ gives

$$f(a+h) \approx f(a) + f'(a)h.$$

for h close to 0. Finally, define $z = a + h$ and substitute to get

$$f(z) \approx f(a) + f'(a)(z-a)$$

for z close to a. Geometrically, we have approximated f near z using a linear function (which is the equation for its tangent line). We call this a *first-order* approximation of f. The idea behind a Taylor polynomial or Taylor series is to get higher-order and more accurate approximations of f.

Here is a heuristic that is useful in understanding where Taylor series come from. Suppose you know that $f(z)$ is given by a power series

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + a_3(z-a)^3 + a_4(z-a)^4 + a_5(z-a)^5 + \cdots$$

Then it follows that

$$\begin{aligned}
f'(z) &= a_1 + 2a_2(z-a) + 3a_3(z-a)^2 + 4a_4(z-a)^3 + 5a_5(z-a)^4 + \cdots \\
f''(z) &= 2\cdot 1\, a_2 + 3\cdot 2\, a_3(z-a) + 4\cdot 3\, a_4(z-a)^2 + 5\cdot 4\, a_5(z-a)^3 + \cdots \\
f^{(3)}(z) &= 3\cdot 2\cdot 1\, a_3 + 4\cdot 3\cdot 2\, a_4(z-a) + 5\cdot 4\cdot 3\, a_5(z-a)^2 + \cdots \\
f^{(4)}(z) &= 4\cdot 3\cdot 2\cdot 1\, a_4 + 5\cdot 4\cdot 3\cdot 2\, a_5(z-a) + \cdots
\end{aligned}$$

$$\vdots$$

Next, evaluate these derivatives at $z = a$, at which point, all terms of the form $(z-a)^k$ vanish, and solve for the a_i:

$$f'(a) = a_1 \quad \Rightarrow \quad a_1 = f'(a)$$

$$f''(a) = 2 \cdot 1\, a_2 \quad \Rightarrow \quad a_2 = \frac{f''(a)}{2!}$$

$$f^{(3)}(a) = 3 \cdot 2 \cdot 1\, a_3 \quad \Rightarrow \quad a_3 = \frac{f^{(3)}(a)}{3!}$$

$$f^{(3)}(a) = 3 \cdot 2 \cdot 1\, a_3 \quad \Rightarrow \quad a_3 = \frac{f^{(3)}(a)}{3!}$$

$$f^{(4)}(a) = 4 \cdot 3 \cdot 2 \cdot 1\, a_4 \quad \Rightarrow \quad a_3 = \frac{f^{(4)}(a)}{4!}$$

$$\vdots$$

Thus, if f is a power series, we can determine the coefficients:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n.$$

Note: By definition, $f^{(0)}(a) = f(a)$, and $f^{(n)}$ is the n-th derivative for $n \geq 1$.

Definition. The *k-th order Taylor polynomial* for f centered at a is

$$\sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(z-a)^n,$$

and the *Taylor series* for f centered at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n.$$

Example. The first-order Taylor polynomial of f is

$$f(a) + f'(a)(z-a),$$

which is the approximation we derived earlier using the definition of the derivative of f.

Example. Consider the polynomial

$$f(z) = 3 - 2z + 3z^2 + z^3$$

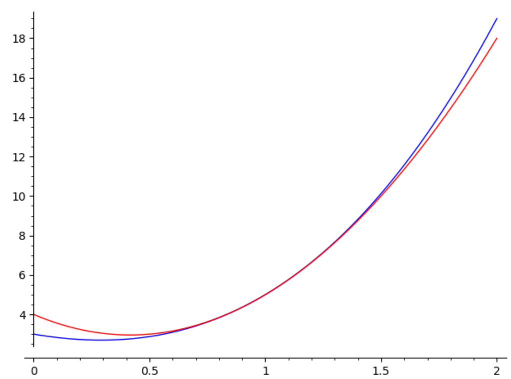
To find the Taylor series centered at $a = 1$, we compute the relevant derivatives:

$$f'(z) = -2 + 6z + 3z^2, \quad f''(z) = 6 + 6z, \quad f^{(3)}(z) = 6,$$

and all higher-order derivatives are 0. The second-order Taylor polynomial of f centered at $a = 1$ is

$$\begin{aligned} T_2(z) &:= f(1) + f'(1)(z-1) + \frac{f''(1)}{2!}(z-1)^2 \\ &= 5 + 7(z-1) + 6(z-1)^2. \end{aligned}$$

Here is a plot of f and T_2 near $z = 1$ (T_2 is in red, and note that the two axes have different scales):



Close to $z = 1$, the second-order Taylor polynomial is a great approximation for f. The third-order Taylor polynomial is

$$\begin{aligned} T_3(z) &:= f(1) + f'(1)(z-1) + \frac{f''(1)}{2!}(z-1)^2 + \frac{f^{(3)}(1)}{3!}(z-1)^3 \\ &= 5 + 7(z-1) + 6(z-1)^2 + (z-1)^3. \end{aligned}$$

Since all derivatives of f of order 4 or higher are 0, the fourth- and higher-order Taylor polynomials are all equal to this third-order Taylor polynomial (as is the Taylor series).

In fact, this third-order Taylor polynomial is as good an approximation to f as you could ever want: you can check for yourself that if you multiply out T_3, you actually get f, i.e., $f(z) = T_3(z)$ for all z. This is what you'd expect from the best third-order approximation of a polynomial of degree 3.

Proposition. Let f be a polynomial of degree d, and let T_k denote the Taylor polynomial for f of order k centered at any point. Then $T_k(z) = f(z)$ for all z for all $k \geq d$.

Example. Let $f(z) = \cos(z)$. The derivatives of f are

$$f'(z) = -\sin(z), \quad f''(z) = -\cos(z), \quad f^{(3)}(z) = \sin(z), \quad f^{(4)}(z) = \cos(z), \ldots$$

From this point on, the higher-order derivative cycle among those we've just calculated. Let's compute the Taylor series at $z = 0$. We have

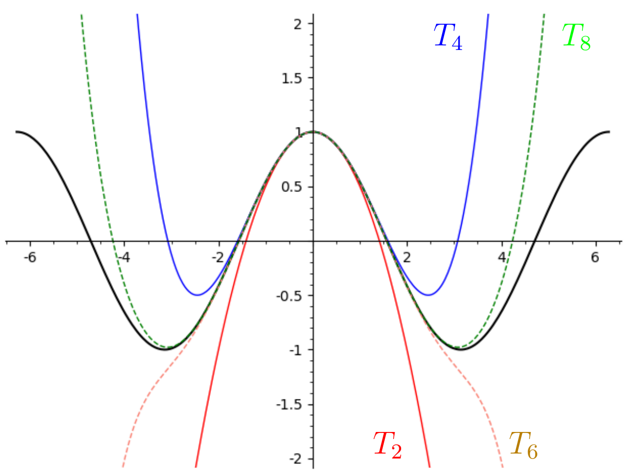
$$f(0) = 1,\ f'(0) = 0,\ f''(0) = -1,\ f^{(3)}(0) = 0,\ f^{(4)}(0) = f(0) = 1,$$

and these values keep cycling. This means that the Taylor series for $\cos(z)$ is

$$T := \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

$$= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots$$

Here is a plot of $f(z) = \cos(z)$ (in black), $T_2(z)$ (in red), $T_4(z)$ (in blue), $T_6(z)$ (salmon and dashed), and $T_8(z)$ (green and dashed):



You should notice that the approximations near $z = 0$ get successively better.

The following theorems help to express that fact that Taylor polynomials and Taylor series are good approximations for functions.

Theorem (Taylor's theorem over $\mathbb{C}$). Let $A \subseteq \mathbb{C}$ and let $f\colon A \to \mathbb{C}$. Suppose that $B(a;r) \subseteq A$ and that f has derivatives of orders $1, \dots, n$ on $B(a;r)$ where $n \geq 1$. Let $T_{n,a}$ be the Taylor polynomial of order n for f centered at a. Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $z \in B(a;\delta)$, then

$$|f(z) - T_{n,a}(z)| < \varepsilon.$$

Theorem. (Taylor's theorem over $\mathbb{R}$) Let $A \subseteq \mathbb{R}$ and let $f\colon A \to \mathbb{R}$. Suppose that A contains an open interval containing the closed interval $[u, v]$. Also suppose that all the derivatives up to order n exist and are continuous on $[u, v]$ and the $(n+1)$-th derivative exists on (u, v). Let $a \in A$, and let $T_{n,a}$ be the Taylor polynomial of order n for f centered at a. Then for all $x \in [u, v]$, there exists a number c strictly between x and a such that

$$f(x) = T_{n,a}(x) + \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}.$$

How to interpret this last theorem. The term

$$\frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1} = f(x) - T_{n,a}(x)$$

is the error in approximating f by its n-th order Taylor polynomial In particular, if $|f^{n+1}|$ is bounded by M on the interval $[u, v]$, then the error is bounded:

$$\left| \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1} \right| \leq \frac{M}{(n+1)!} |x-a|^{n+1}.$$

Note that if x is close to a then the factor $\frac{1}{(n+1)!} \cdot (x-a)^{n+1}$ will be very small. (For instance, for any constant α, we have $\lim_{n\to\infty} \frac{\alpha^{n+1}}{(n+1)!} = 0$.)