Math 112 lecture for Wednesday, Week 11

Limits of functions

(Supplemental reading: Sections 4.1 and 4.2 in Swanson.)

We now switch our focus from limits of sequences and series to limits of *functions*. Let $F = \mathbb{R}$ or $\mathbb{C}$.

Recall that the definition of the derivative of a function f at a point a looks something like this:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Here we are taking the limit of the function

$$g(h) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

and that function is not defined at exactly the point of interest, i.e., at $h = 0$:

$$g(0) = \frac{f(a+0) - f(a)}{0} = \frac{0}{0} = \text{ undefined.}$$

In this way, a main use of the limit of a function is to determine what value a function should have at a point at which it is not defined. We are able to make this determination since the function is nicely behaved at all nearby points. The following definition characterizes the types of points, called *limit points*, at which we might hope to compute a limit.

Definition. Let $A \subseteq F$. A point $x \in F$ is a *limit point* of A if every open ball centered at x contains a point of A not equal to x. In other words, for all $r > 0$ there exists $y \in B(x, r) \cap A$ such that $x \neq y$.

Roughly, a limit point x of a set A can be approximated arbitrary closely by points besides x that are contained in A. The limit point, itself may or may not be in A.

Examples.

(a) The limit points of $A = (0, 1) \subset \mathbb{R}$ are all points in the closed interval $[0, 1]$. Note that 0 and 1 are limit points of A that are not in A.

(b) The limit points of $A = (0, 1) \cup \{7\} \subset \mathbb{R}$ are again the points of $[0, 1]$. That's because there is an open interval (ball) about 7 that contains no points of A besides 7 (for instance, the interval $(6, 8)$ contains no points in A besides 7). We naturally call 7 an *isolated point* of A.

(c) The set of limit points of the "punctured ball" $B(0;1) \setminus 0$ of radius 1 centered at the origin in $\mathbb{C}$ is the closed ball

$$\overline{B(0;1)} := \{z \in \mathbb{C} : |z| \leq 1\}.$$

Every point in the punctured open ball is a limit point, but so are the points on the boundary and the origin.

Definition. Let $F = \mathbb{R}$ or $\mathbb{C}$, as usual. Let $A \subseteq F$ and $f\colon A \to F$. Let $a \in F$ be a limit point of A. Then the *limit of $f(x)$ as x approaches a is* $L \in F$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then

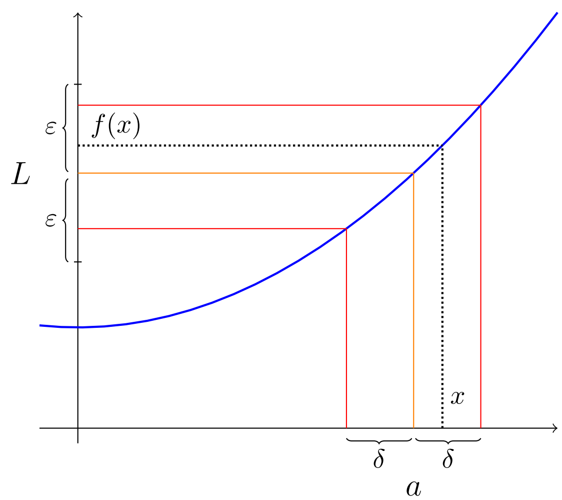
$$|f(x) - L| < \varepsilon.$$

If the limit is L, then we write $\lim_{x \to a} f(x) = L$.

Remark. If you wanted to pack the definition of $\lim_{x \to a} f(x) = L$ into symbols, you could write

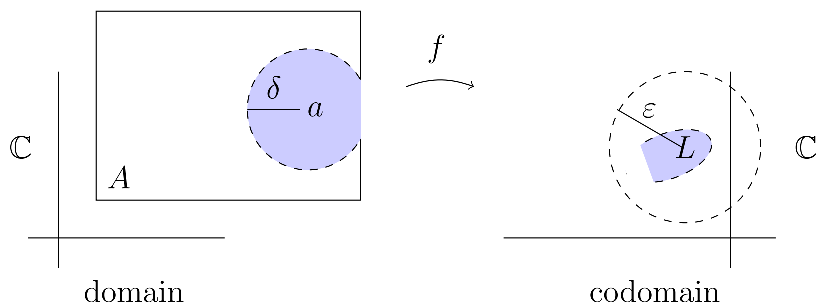
$$\forall \varepsilon > 0, \exists \delta > 0, \text{ s.t. } f\left((B(a;\delta) \setminus \{a\}) \cap A\right) \subseteq B(L;\varepsilon).$$

In the case where $F = \mathbb{R}$, we can interpret the definition of the limit using the graph of f:



Meaning of ε and δ in the definition of the limit when $F = \mathbb{R}$.

Consider now the case where the domain and codomain of f are in $\mathbb{C}$. The graph of f consist of points $(z, f(z))$ where z and $f(z)$ are in $\mathbb{C}$. So to picture the graph we would need four dimensions—two for z and two for $f(z)$. An alternative is to use two copies of the plane, $\mathbb{C}$, and picture the domain and codomain separately. We then try to picture how f moves points in the domain over to points in the codomain. With this set-up, here is the relevant picture for understanding the limit definition:



On the left, we have the codomain A of f, and the blue shaded region is the intersection of an open ball of radius δ with A. Note that for the purposes of the definition of the limit, we should remove the center of the ball, a. On the right, we have the ε-ball about L as the "target", and the blue shaded region is the image of the blue shaded region on the left. The picture shows that for the chosen value of ε, we were able to find a suitable δ.

Remarks.

- We are interested in the behavior of the function f near the point a, but not exactly at the point a. In fact, f need not even be defined at a. For example, consider the function
$$f(x) = \frac{x^2 - x}{x}.$$
If we try to evaluate f at 0, we get $f(0) = \frac{0}{0}$, which does not makes sense (you can't divide by 0), i.e., $\frac{0}{0}$ is not a number. However, the limit is exists at $x = 0$ (and is equal to -1).

- When you see the absolute values in the definition, you should think "distance". The *distance* between the numbers u and v is $|u - v|$. So you should translate $|f(x) - L| < \varepsilon$ as "the distance between $f(x)$ and L is less than ε".

- Consider the part of the definition that says $0 < |x - a| < \delta$. If the expression had just been $|x - a| < \delta$, without the "$0 <$" part, the requirement would be that the distance between the number x and a is less than δ. What does $0 < |x - a|$ add? The only way the absolute value of a number such as $x - a$ can be 0 is if $x - a = 0$ or, equivalently, $x = a$. Thus, requiring $0 < |x - a|$ is just requiring that x not equal a. This is just what we need since, after all, the function f may not be defined at a.

- Note the *quantifiers* "for all" and "there exists" in the definition. Just as with the definition of limits of sequences, it takes a while to appreciate their importance, but they are essential. First take the "for all" part. The definition say that for all $\varepsilon > 0$, we are going to want $|f(x) - L| < \varepsilon$. Translating: for all $\varepsilon > 0$, we will want to make the distance between $f(x)$ and L less that $\varepsilon > 0$. Our goal is to make f close to L, and the ε is a measure of how close. By making ε small and requiring $|f(x) - L| < \varepsilon$, we are ensuring that $f(x)$ is within a distance of ε from L.

  Next, consider the "there exists" part of the definition. It says that if you want $f(x)$ to be within a distance of ε of L, then it suffices to make $0 < |x - a| < \delta$. In other words, you can to make x within a distance of δ of a (remembering that we don't care what happens when $x = a$).

  Given any $\varepsilon > 0$ (a challenge to make $f(x)$ close to L), you want to find an appropriate distance $\delta > 0$ (so that if x is δ-close to a, then $f(x)$ is ε-close to L).

  With sequences, the game was: given ε find N. With functions, the game becomes: given ε, find δ.

Warning. At some point, in proving a statement of the form $\lim_{x\to a} f(x) = L$, you will be tempted to have δ be a function of x. That is not allowed! (On the other hand, δ is typically a function of ε, just as with sequences, N is typically a function of ε (but cannot be a function of n).)

Example. Consider the function

$$\begin{aligned} f\colon F &\to F \\ x &\mapsto 5x + 3. \end{aligned}$$

Then $\lim_{x\to 2} 5x + 3 = 13$.

Proof. Given $\varepsilon > 0$, let $\delta = \varepsilon/5$. Then if

$$0 < |x - 2| < \delta,$$

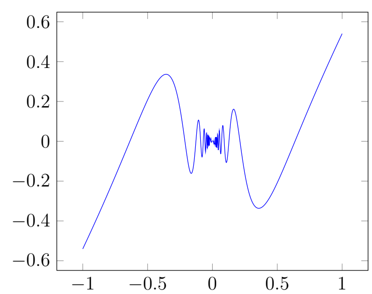
it follows that

$$\begin{aligned} |f(x) - 13| &= |(5x + 3) - 13| \\ &= |5x - 10| \\ &= 5|x - 2| \end{aligned}$$

$$
\begin{aligned}
&< 5\delta \\
&= 5 \cdot \frac{\varepsilon}{5} \\
&= \varepsilon.
\end{aligned}
$$

□

Example. Prove that $\lim_{x\to 0} x\cos(1/x) = 0$.



Graph of $f(x) = x\cos(1/x)$.

Proof. Given $\varepsilon > 0$, let $\delta = \varepsilon$. Suppose that $0 < |x - 0| < \delta$; in other words, suppose that $0 < |x| < \varepsilon$. Then, since $|\cos(y)| \leq 1$ for all y, we have

$$
\begin{aligned}
|x\cos(1/x)| &= |x||\cos(1/x)| \\
&\leq |x| \\
&< \delta \\
&= \varepsilon.
\end{aligned}
$$

□

Example. Proving limit statements for simple function like $f(x) = x^2$ can be a challenge. For instance, here we prove the "obvious" statement that $\lim_{x\to 5} x^2 = 25$. This example points out the need for better tools (e.g., a new-from-old limit theorem like we had with sequences) allowing us to avoid going down to the level of using ε-δ arguments.

Proof. Given $\varepsilon > 0$, let $\delta = \min\{1, \varepsilon/11\}$, i.e., δ is the minumum of 1 and $\varepsilon/11$. So $\delta \leq 1$ and $\delta \leq \varepsilon/11$ (with equality holding in at least one of these). Suppose that x satisfies $0 < |x - 5| < \delta$. Since $\delta \leq 1$, the distance between x and 5 is less than 1. It follows $4 < x < 6$, and hence, adding 5 across this string of inequalities, we get $9 < x + 5 < 11$. In particular, $|x + 5| < 11$. Therefore,

$$|x^2 - 25| = |(x + 5)(x - 5)| = |x + 5||x - 5| < 11|x - 5|.$$

Now, since $\delta \le \varepsilon/11$ and $|x - 5| < \delta$, it follows that

$$|x^2 - 25| < 11|x - 5| < 11 \cdot \frac{\varepsilon}{11} = \varepsilon,$$

as required. □

Example. Claim: $\lim_{x\to 16} \sqrt{x} = 4$.

Proof. Given $\varepsilon > 0$, let $\delta = 4\varepsilon$, and suppose that

$$0 < |x - 16| < \delta = \frac{\varepsilon}{4}.$$

Then

$$\begin{aligned} |\sqrt{x} - 4| &= \left|(\sqrt{x} - 4) \cdot \frac{\sqrt{x} + 4}{\sqrt{x} + 4}\right| \\ &= \frac{|x - 16|}{|\sqrt{x} + 4|} \\ &< \frac{|x - 16|}{4} \\ &= \frac{1}{4}\,|x - 16| \\ &< \frac{1}{4}\,\delta \\ &= \varepsilon. \end{aligned}$$

□