Math 112 lecture for Monday, Week 11

SERIES TESTS III

(Supplemental reading: Section 9.2 in Swanson.)

We continue our discussion of the standard tests for determining whether a series converges:

1. the geometric series test
2. the n-th term test
3. the comparison test
4. the limit comparison test
5. the alternating series test
6. the absolute convergence test
7. **the ratio test**
8. **the root test**
9. **the integral test**
10. **the p-series test**.

7. The ratio test.

Proposition (ratio test). Let $\{a_n\}$ be a sequence of positive real numbers, and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = R.$$

Then

(a) if $R < 1$, then $\sum a_n$ converges;

(b) if $R > 1$ or $R = \infty$, then $\sum a_n$ diverges;

(c) if $R = 1$, the test is inconclusive.

Example. The ratio test shows that $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges since, as $n \to \infty$,

$$\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{n!}{(n+1)!} = \frac{1}{n+1} \longrightarrow 0 < 1.$$

Example. To see that the case $R = 1$ is inconclusive, consider the series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$. The former series diverges and the latter converges, yet we have

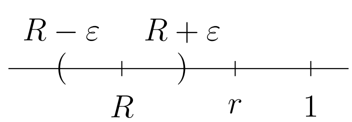
$$\lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = 1.$$

Example. Even if a series does not have positive terms, one can use the ratio test to consider if a series is absolutely convergent. For instance, we can see that the series $\sum_{n=}^{\infty}(-1)^n n\left(\frac{1}{2^n}\right)$ is absolutely convergent (hence, convergent), since

$$\lim_{n\to\infty} \frac{(n+1)\left(\frac{1}{2^{n+1}}\right)}{n\left(\frac{1}{2^n}\right)} = \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{1}{2} = \frac{1}{2} < 1.$$

Note. The tell-tale sign that the ratio test might apply is the presence of factorials and exponents.

Proof of the ratio test. First suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = R < 1$. Fix any real number r such that $0 \leq R < r < 1$. Our goal, roughly, is to apply the comparison test to $\sum a_n$, comparing it with the convergent geometric series $\sum r^n$. Applying the definition of the limit, we can find N such that $n > N$ implies $|R - \frac{a_{n+1}}{a_n}| < \varepsilon$ where ε has been choose small enough so that this condition forces $\frac{a_{n+1}}{a_n} < r$:



Thus, $n > N$ implies that $0 \leq \frac{a_{n+1}}{a_n} < r$, i.e.,

$$0 \leq a_{n+1} < a_n r.$$

In particular, taking $n = N + 1$, this means

$$a_{N+2} < a_{N+1} r$$

and then

$$a_{N+3} < a_{N+2} r < a_{N+1} r^2.$$

Continuing,

$$a_{N+4} < a_{N+3} r < a_{N+1} r^3,$$

and so on. Letting $c := a_{N+1}$, we may show by induction that

$$a_{N+k} < cr^{k-1}$$

for $k \geq 2$. Since $|r| = r < 1$, the series

$$\sum_{k=2}^{\infty} cr^{k-1} = cr \sum_{k=0}^{\infty} r^k$$

converges. By the comparison test, the series $\sum_{n=N+2}^{\infty} a_n$ converges. Since this is a tail of our original series $\sum a_n$, the original series converges.

Now suppose that $R > 1$ or $R = \infty$. Then we can take N such that

$$n > N \quad \Rightarrow \quad \frac{a_{n+1}}{a_n} > 1 \qquad a_{n+1} > a_n.$$

By transitivity of $\geq$, we have $a_n \geq a_{N+1} > 0$ for all $n > N$. Thus, $\lim_{n\to\infty} a_n \neq 0$, and the series diverges by the n-th term test. □

8. The root test. We will not discuss this test, but include it here as one of the standard tests.

Proposition (root test). Let $\sum a_n$ be a series of nonnegative terms. Suppose that $\lim a_n^{1/n} = b$. If $b < 1$, the series converges. If $b > 1$, the series diverges. If $b = 1$, the test is inconclusive.

9. The integral test.

Proposition (integral test). Suppose $f(x)$ is a continuous, positive, decreasing function whose domain contains $(0, \infty)$. Then $\sum f(n)$ converges if and only if $\lim_n (\int_1^n f(x)\,dx)$ converges, i.e., if and only if $\int_1^{\infty} f(x)\,dx$ converges.

Example. We can use the integral test to give a quick proof that the harmonic series diverges:

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{n\to\infty} \int_1^n \frac{1}{x}\,dx = \lim_{n\to\infty} \ln x\Big|_1^n = \lim_{n\to\infty} (\ln n - \ln 1) = \lim_{n\to\infty} \ln n = \infty.$$

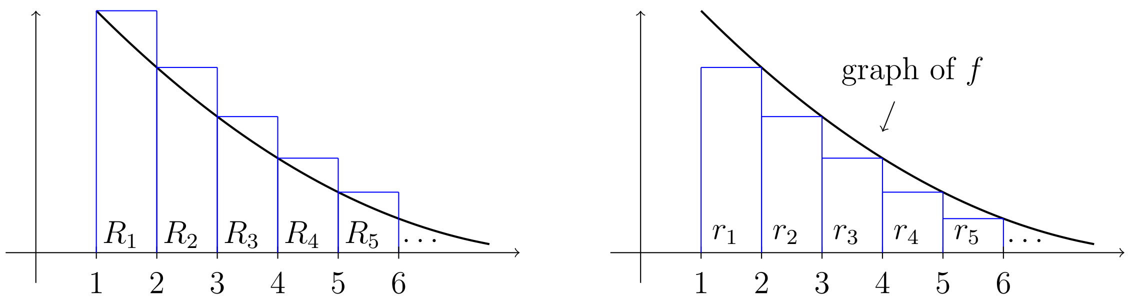
Thus, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the integral test.

Similarly, we may show that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges:

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{n\to\infty} \int_1^n \frac{1}{x^2}\,dx = \lim_{n\to\infty} -\frac{1}{x}\Big|_1^n = \lim_{n\to\infty} (-\frac{1}{n} + 1) = 1.$$

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the integral test.

Proof of the integral test. Consider the standard pictures for the upper and lower Riemann sums for the integral of f:



upper sum lower sum

For each $n = 1, 2, \dots,$ we have

$$\text{area}(R_n) = \text{base}(R_n) \cdot \text{height}(R_n) = 1 \cdot f(n) = f(n). \tag{1}$$

Similarly,

$$\text{area}(r_n) = \text{base}(r_n) \cdot \text{height}(r_n) = 1 \cdot f(n+1) = f(n+1). \tag{2}$$

Further, since f is decreasing and nonnegative

$$\text{area}(r_n) \le \int_n^{n+1} f(x)\,dx \le \text{area}(R_n),$$

and thus, using (1) and (2),

$$f(n+1) \le \int_n^{n+1} f(x)\,dx \le f(n). \tag{3}$$

Summing, we get

$$\sum_{n=1}^{k} f(n+1) \le \sum_{n=1}^{k} \int_n^{n+1} f(x)\,dx \le \sum_{n=1}^{k} f(n).$$

Then note that

$$\sum_{n=1}^{k} \int_n^{n+1} f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \cdots + \int_k^{k+1} f(x)\,dx = \int_1^{k+1} f(x)\,dx$$

since we are just adding areas under the graph of f. So the result follows from the ordinary comparison theorem applied to (3). $\square$

10. The p-series test.

Proposition (p-series test). Let $p \in \mathbb{R}$. Then the series

$$\sum \frac{1}{n^p}$$

converges if and only if $p > 1$.

Example. The p-series tests says these series converge:

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^{1.000001}}$$

and these series do not:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^{0.999999}}.$$

Proof of the p-series test. Apply the integral test. We did the case $p = 1$ earlier. So assume $p \neq 1$. In that case, we get

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x}\, dx &= \int_1^{\infty} x^{-1}\, dx \\
&= \lim_{n \to \infty} \int_1^n x^{-1}\, dx \\
&= \lim_{n \to \infty} \frac{1}{1-p} x^{1-p} \Big|_1^n \\
&= \frac{1}{1-p} \lim_{n \to \infty} (n^{1-p} - 1) \\
&= \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \infty & \text{if } p < 1. \end{cases}
\end{aligned}$$

□

Interesting question. Does the following sum converge:

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}} \; ?$$

The p-series test does not apply since the exponent is not constant.